CONVERGENCE ANALYSIS OF NEWTON-COTES METHODS: OPTIMIZING SUB-INTERVALS SELECTION FOR PRECISE INTEGRAL APPROXIMATION

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ABSTRACT

This study explored the piecewise approach of the closed Newton-Cotes quadrature formulas (Trapezoidal, Simpson's 1/3, and 3/8 rules) and how well they work with different kinds of functions in terms of convergence and accuracy. MATHEMATICA software was used to approximate the integrals and determine their errors, allowing for a comparison of convergence and accuracy. Simpson's 1/3 and 3/8 rules consistently outperformed the trapezoidal rule, demonstrating faster convergence and greater accuracy across a wide range of functions. However, as tolerance levels increased to a considerable magnitude, Simpson's 3/8 rule emerged as the most robust among the three methods. We recommend investigating various domains to substantiate the findings of this study including a comprehensive error analysis that includes truncation error, round-off error, and error bounds to provide a more detailed understanding of the sources and magnitude of errors and to include higher-dimensional integrals to provide valuable insights into the robustness of these methods.

KEYWORDS

Numerical Quadrature, Newton-Cotes, Trapezoidal Rule, Simpson's 1/3 rule, Simpson's 3/8 rule

1. INTRODUCTION

The Newton-Cotes quadrature methods are a class of numerical integration techniques based on polynomial interpolation over a set of equally spaced nodes [1]. These techniques provide a direct approximation of definite integral $\int_a^b f(x)\,dx$. However, they can quickly become unstable when applied to functions with distinct characteristics specially those that are not well-approximated by low-degree polynomials over large intervals. This leads to a substantial increase of error, as the polynomial interpolant may not accurately capture the behaviour of the integrand over a large interval.

The Newton-Cotes formulas are obtained by interpolating polynomials which approximate the tabulated function $f(x)$. Then integrating the function $f(x)$ over some interval $[a, b]$ divided into n equal sub-intervals such that $f_n = f(x_n)$ and $h \equiv \frac{b-a}{n}$ [2]. The two types of Newton-Cotes quadrature methods are open and closed. The $(n + 1)$-point closed Newton-Cotes method includes endpoints of the closed interval $[a, b]$ as nodes. It uses nodes $x_i = x_0 + ih \;\; for\ i = 0, 1, 2, \ldots\ldots n,\ where\ x_0 = a,\ x_n = b\ and\ h = \frac{b-a}{n}$ [3]. Unlike closed Newton-Cotes, open Newton-Cotes formulas do not include the endpoints of $[a, b]$.

The most recognized closed Newton-Cotes formulas are the trapezoidal rule and Simpson's 1/3 and 3/8 rules, mainly because of their balance of simplicity and preciseness. These methods are widely applied across various areas of study, such as engineering, economics and finance, computational science, and more [4]. Trapezoidal rule and Simpson's rule are interpolatory quadrature based on linear and quadratic interpolants, respectively [5]. These low-order rules are often slow to converge and return inaccurate results over large intervals due to the oscillatory nature of high-degree polynomials.

High-order quadrature techniques would be necessary to integrate high-degree polynomials, however the values of the coefficients in these techniques are difficult to obtain and often numerically unstable [1][3]. Alternatively, the Composite Quadrature rules for trapezoidal and Simpson's are simpler approach and considered an effective way to increase the accuracy of the result when integrating high-degree polynomials. These rules involve partitioning the interval $[a, b]$ into subintervals then using low-order interpolants (e.g. Trapezoidal and Simpson) to approximate the integrals in each subinterval.

The composite approach in numerical quadrature, particularly, using low-order Newton-Cotes methods like the trapezoidal rule and Simpson's rule has been extensively studied due to its practical advantages in improving accuracy and managing the convergence of integral approximations. The study by Udin, M.J.et al evaluated and compared the performance of Trapezoidal, Simpson's 1/3, and 3/8 rules in terms of accuracy and efficiency, utilizing error analysis to determine which method performs better in providing accurate results [6]. Daan Huybrechs, reviewed and presented the difference between the low-order variants of the class of Newton-Cotes quadrature and the high-order quadrature (Least-Square quadrature) in terms of numerical stability and convergence properties [1]. Yalda Qani used the advanced family of closed Newton–Cotes numerical composite formulas to demonstrate some of the computational capabilities of the Maple package [7]. In the A.H. Nzokem study, he implemented both analytical and numerical methods (composite Newton-Cotes quadrature formulas) in solving the Gamma Distribution Hazard function [8]. Sheehan Olver investigated and explored the numerical integration of highly oscillatory functions, over both univariate and multivariate domains, using the combination of Filon-type methods and Levin-type methods [9]. Furthermore, Hamid Mottaghi Golshan proposed a numerical iterative method based on Picard iterations and Newton-Cotes rules that can be used to approximate the multidimensional Fredholm–Urysohn integral equation of the second kind [10]. The study of Magalhaes, P. et al. discussed and compared the closed and open Newton-Cotes quadrature formula, utilizing twenty segments to assess the accuracy and convergence of the two methods [11]. In the work of Ali, A. J. et al, they utilized error and stability analysis to evaluate the degree of accuracy of analytical, numerical (Trapezoidal and Simpson's 1/3 and 3/8 rules), and software-assisted (MatLab) methods [12]. Likewise, Erme Sermutlu conducted a study utilizing MatLab and error analysis to compare Newton-Cotes and Gauss quadrature methods over the same number of intervals for diverse types of functions [13]. Additionally, John Roumeliotis used the software-assisted method (Maple) to prove the Ostrowski and corrected Trapezoidal inequalities and stated two new fourth-order quadrature formula [14]. Wu, J et. Al. explored and used the composite Newton-Cotes methods for computation of Hadamard finite-part integral with the second order singularity focusing on their pointwise superconvergence properties [15]. And, in the study conducted by Clarence Burg, he presented a new closed Newton–Cotes type of quadrature formula that uses first and higher order derivatives to increase the order of accuracy of the numerical approximations of definite integrals [16].

This paper, like the cited studies, aims to investigate the Newton-Cotes quadrature methods (Trapezoidal and Simpson's rules) for approximating definite integrals and analyse their convergence and accuracy. However, it will focus on the piecewise approach (composite rules) of the closed Newton-Cotes quadrature methods and explore their applicability to various special function types. It will also determine how the selection of sub-intervals affects the accuracy and convergence of these approaches.

2. MATERIALS AND METHODS

Presented in this section is the piecewise approach of the three commonly used closed Newton-Cotes quadrature formulas.

2.1. Closed Newton-Cotes Quadrature Formulas

Newton–Cotes methods consist of approximating the integrand f by a polynomial of degree n, which matches f at n evenly spaced nodes [10]. The $(n+1)$-point closed Newton-Cotes method includes endpoints of the closed interval $[a,b]$ as nodes [3, p.198]. It uses nodes $x_i = x_0 + ih \; for \; i = 0, 1, \dots, n, \; where \; x_0 = a, \; x_n = b \; and \; h = \frac{b-a}{n}$ assuming the form

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} a_i f(x_i), \tag{2.1}$$

where the weight

$$a_i = \int_{x_0}^{x_n} L_i(x)dx = \int_{x_0}^{x_n} \prod_{j=0}^{n} \prod_{j\neq i} \frac{(x - x_j)}{(x_i - x_j)} dx. \tag{2.2}$$

Using the first order Lagrange interpolating polynomial, (2.1) and (2.2) return

$$\int_a^b f(x)dx \approx a_0 f(x_0) + a_1 f(x_1)$$

with,

$$a_0 = \int_{x_0}^{x_1} \left(\frac{x - x_1}{(x_0 - x_1)}\right) dx = \frac{1}{2}(x_1 - x_0),$$

and

$$a_1 = \int_{x_0}^{x_1} \left(\frac{x - x_0}{x_1 - x_0}\right) dx = \frac{1}{2}(x_1 - x_0).$$

Combining and simplifying, we obtain the standard trapezoidal rule,

$$\int_a^b f(x)dx \approx \frac{x_1 - x_0}{2}\big(f(x_0) + f(x_1)\big). \tag{2.3}$$

Since $n = 1, \; h = b - a, \; x_0 = a, and \; x_1 = b$, (2.3) can also be expressed as

$$\int_a^b f(x)dx \approx \frac{h}{2}\big(f(a) + f(b)\big).$$

In a similar manner, using the second-order Lagrange interpolating polynomials, (2.1) and (2.2) give

$$\int_a^b f(x)dx \approx a_0 f(x_0) + a_1 f(x_1) + a_2 f(x_2)$$

with,

$$a_0 = \int_{x_0}^{x_2} \left(\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \right) dx = \frac{1}{6}(x_2 - x_0),$$
$$a_1 = \int_{x_0}^{x_2} \left(\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \right) dx = \frac{2}{3}(x_2 - x_0),$$
$$a_2 = \int_{x_0}^{x_2} \left(\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \right) dx = \frac{1}{6}(x_2 - x_0).$$

We obtain the standard 1/3 Simpson's rule, which is one of the most recognized closed Newton-Cotes quadrature approaches in practice.

$$\int_a^b f(x)dx \approx \frac{x_2 - x_0}{6}\big(f(x_0) + 4f(x_1) + f(x_2)\big). \tag{2.4}$$

Since $n = 2,\ h = \frac{b-a}{2},\ x_0 = a,\ x_1 = \frac{a+b}{2},\ and\ x_2 = b,$ (2.4) can also be written as

$$\int_a^b f(x)dx \approx \frac{h}{3}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right).$$

Likewise, utilizing the third-order Lagrange interpolating polynomial (2.1) becomes,

$$\int_a^b f(x)dx \approx a_0 f(x_0) + a_1 f(x_1) + a_2 f(x_2) + a_3 f(x_3),$$

with

$$a_0 = \int_{x_0}^{x_3} \left(\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \right) dx$$
$$= \left(\frac{1}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \right) \Big[\Big(\tfrac{1}{4}(x_3^4) - \tfrac{1}{3}(x_1 + x_2 + x_3)\, x_3^3 + \tfrac{1}{2}(x_1x_2 + x_1x_3 + x_2x_3)\, x_3^2 - x_1x_2x_3^2 \Big) - \Big(\tfrac{1}{4}x_0^4 - \tfrac{1}{3}(x_1 + x_2 + x_3)\, x_0^3 + \tfrac{1}{2}(x_1x_2 + x_1x_3 + x_2x_3)\, x_0^2 - x_0x_1x_2x_3 \Big) \Big],$$

$$a_1 = \int_{x_0}^{x_3} \left(\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \right) dx$$
$$= \left(\frac{1}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \right) \Big[\Big(\tfrac{1}{4}x_3^4 - \tfrac{1}{3}(x_0 + x_2 + x_3)\, x_3^3 + \tfrac{1}{2}(x_0x_2 + x_0x_3 + x_2x_3)\, x_3^2 - x_0x_2x_3^2 \Big) - \Big(\tfrac{1}{4}x_0^4 - \tfrac{1}{3}(x_0 + x_2 + x_3)\, x_0^3 + \tfrac{1}{2}(x_0x_2 + x_0x_3 + x_2x_3)\, x_0^2 - x_0^2x_2x_3 \Big) \Big],$$

$$a_2 = \int_{x_0}^{x_3} \left(\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \right) dx$$

$$= \left(\frac{1}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)}\right)\left[\left(\frac{1}{4}x_3^4 - \frac{1}{3}(x_0 + x_1 + x_3)x_3^3 + \frac{1}{2}(x_0x_1 + x_0x_3 + x_1x_3)x_3^2 - x_0x_1x_3^2\right) - \left(\frac{1}{4}x_0^4 - \frac{1}{3}(x_0 + x_1 + x_3)x_0^3 + \frac{1}{2}(x_0x_1 + x_0x_3 + x_1x_3)x_0^2 - x_0^2x_1x_3\right)\right],$$

$$a_3 = \int_{x_0}^{x_3} \left(\frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}\right) dx$$

$$= \left(\frac{1}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}\right)\left[\left(\frac{1}{4}x_3^4 - \frac{1}{3}(x_0 + x_1 + x_2)x_3^3 + \frac{1}{2}(x_0x_1 + x_0x_2 + x_1x_2)x_3^2 - x_0x_1x_2x_3\right) - \left(\frac{1}{4}x_0^4 - \frac{1}{3}(x_0 + x_1 + x_3)x_0^3 + \frac{1}{2}(x_0x_1 + x_0x_3 + x_1x_3)x_0^2 - x_0^2x_1x_2\right)\right].$$

Combining and simplifying, we obtain the 3/8 Simpson's rule

$$\int_a^b f(x)dx \approx \frac{3h}{8}\left(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right). \tag{2.5}$$

Since $n = 3,\ h = \frac{b-a}{3},\ x_0 = a,\ x_1 = \frac{2a+b}{3},\ x_2 = \frac{a+2b}{3},\ and\ x_3 = b,$ (2.5) can be expressed as

$$\int_a^b f(x)dx \approx \frac{3h}{8}\left(f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right).$$

2.2. Measure of Exactness

Definition 2.1 [5, p. 121] An interpolatory quadrature rule has *degree of exactness* (degree of precision) m if for all $f \in P_m$ (polynomial interpolant),

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} a_i f(x_i).$$

The definition indicates that a degree-n quadrature method has a degree of exactness $m \geq n$. This further implies that it can exactly integrate $f \in P_n$. However, there are exceptional cases wherein a degree-n interpolant can also exactly integrate higher degree polynomials. Consequently, the degree of precision of trapezoidal rule and Simpson's rules are one and three, respectively.

2.3. Error Analysis for Closed Newton-Cotes Quadrature Methods

The following theorem serves as basis for error analysis of the closed Newton-Cotes quadrature formulas since they are founded on polynomial interpolation and weighted sum of function values at specific nodes.

Theorem 2.1 [5, p. 217] (Interpolation Error Formula) Suppose $f \in C^{n+1}[a,b]$ and let $p_n \in P_n$ denote the polynomial that interpolates $\{x_i, f(x_i)\}_{i=0}^{n}$ for distinct points $x_i \in [a,b], i = 0, 1, \ldots, n$. Then for every $x_i \in [a,b]$ there exist $\xi \in [a,b]$ such that

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} |x - x_i|. \tag{2.6}$$

From this formula follows a bound for the worst error over $[a,b]$:

$$max_{x \in [a,b]} |f(x) - p_n(x)| \leq \left(max_{\xi \in [a,b]} \frac{|f^{(n+1)}(\xi)|}{(n+1)!} \right) \left(max_{x \in [a,b]} \prod_{i=0}^{n} |x - x_i| \right).$$

Proof. Let x be some arbitrary points in the interval $[a,b]$ and let $E(f) = f(x) - p_n(x)$ be the interpolation error of $f\ at\ (x)$. We want an expression to describe $E(f)$. If $x = x_i$ for any $i \in \{0, 1, \ldots, n\}$ then $E(f) = 0$ and choosing arbitrary ξ in (a,b) returns (2.6).

Now, suppose $x \neq x_i$ for all $i \in \{0, 1, \ldots, n\}$, to describe $E(z)$ we define a function g for z in $[a,b]$ by

$$g(z) = f(z) - p_n(z) - \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (z - x_i).$$

Since $f \in C^{n+1}[a,b]$ and $p_n \in P_n$, it follows that $g \in C^{n+1}[a,b]$. For $z = x_i$, we obtain

$$g(x_i) = f(x_i) - p_n(x_i) - \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x_i - x_i) = 0.$$

Furthermore,

$$g(x) = f(x) - p_n(x) - \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i) = 0.$$

Hence, $g \in C^{n+1}[a,b]$, and $g = 0$ at the $n+2$ distinct numbers $x_0, x_1, \ldots, x_n$. By Rolle's theorem there exist a number ξ in (a,b) for which $g^{(n+1)}(\xi) = 0$. And the derivative of $g^{(n+1)}(\xi)$ is

$$g^{(n+1)}(z) = f^{(n+1)}(z) - {p_n}^{(n+1)}(t) - \frac{d^{(n+1)}}{dz^{(n+1)}} \left[\frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (z - x_i) \right].$$

Since $P(x)$ is of degree at most n, then the $(n+1)$ derivative $P^{(n+1)}(x) = 0$. Additionally, $\prod_{i=0}^{n} (z - x_i)$ is of degree $(n+1)$ polynomial, thus

$$\frac{d^{(n+1)}}{dz^{(n+1)}}\left[\frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}(z-x_i)\right]_{t=\xi} = \frac{f^{(n+1)}(\xi)}{(n+1)!}\cdot\frac{d^{(n+1)}}{dz^{(n+1)}}\left[\prod_{i=0}^{n}(z-x_i)\right]$$
$$= \frac{f^{(n+1)}(\xi)}{(n+1)!}\cdot(n+1)!$$
$$= f^{(n+1)}(\xi).$$

Equation (2.6.1) becomes,

$0 = f^{(n+1)}(z) - 0 - f^{(n+1)}(\xi)$

$f^{(n+1)}(z) = f^{(n+1)}(\xi).$

Therefore, for some ξ $[a, b]$

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}|x - x_i|.$$

Theorem 2.2 [3, p. 198] Suppose that $\sum_{i=0}^{n} a_i f(x_i)$ denotes The $(n+1)$-point closed Newton-Cotes formula with $x_0 = a,\ x_n = b\ and\ h = \frac{b-a}{n}$. There exist $\xi \in (a, b)$ for which

$$\int_a^b f(x)dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+3} f^{(n+2)}(\xi)}{(n+2)!}\int_0^n z^2\,(z-1)\ldots.(z-n)dz, \tag{2.7}$$

If n is even and $f \in C^{n+2}[a, b]$, and

$$\int_a^b f(x)dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+2} f^{(n+1)}(\xi)}{(n+1)!}\int_0^n z\,(z-1)\ldots.(z-n)dz, \tag{2.8}$$

If n is odd and $f \in C^{n+1}[a, b]$.

Using this theorem with $n = 1$, (2.8) returns

$$\int_a^b f(x)dx = \sum_{i=0}^{1} a_1 f(x_1) + \frac{h^3 f^{(2)}(\xi)}{2!}\int_0^1 z\,(z-1)dz.$$

From (2.3) with $x_0 = a,\ x_1 = b,\ and\ h = b - a,$ we obtain the standard trapezoidal rule with its error term,

$$\int_a^b f(x)dx = \frac{h}{2}\big(f(x_0) + f(x_1)\big) + \frac{h^3 f^{(2)}(\xi)}{2!}\int_0^1 z\,(z-1)dz$$
$$= \frac{h}{2}\big(f(x_0) + f(x_1)\big) + \frac{h^3 f^{(2)}(\xi)}{2!}\left(-\frac{1}{6}\right)$$
$$\int_a^b f(x)dx = \frac{h}{2}\big(f(x_0) + f(x_1)\big) - \frac{h^3 f^{(2)}(\xi)}{12} \tag{2.9}$$

where $x_0 < \xi < x_2.$

Similarly, we obtain the standard Simpson's 1/3 rule with its error term using (2.4) with $n = 2,\ x_0 = a,\ x_2 = b,\ h = \frac{b-a}{2}$ and (2.7) becomes,

$$\int_a^b f(x)dx = \frac{h}{6}\big(f(x_0) + 4f(x_1) + f(x_2)\big) + \frac{h^5 f^{(4)}(\xi)}{4!}\int_0^2 z\,(z-1)dz$$

$$= \frac{h}{6}\left(f(x_0)+4f(x_1)+f(x_2)\right)+\frac{h^5 f^{(4)}(\xi)}{4!}\left(-\frac{4}{15}\right)$$

$$\int_a^b f(x)dx = \frac{h}{6}\left(f(x_0)+4f(x_1)+f(x_2)\right)-\frac{h^5 f^{(4)}(\xi)}{90} \qquad (2.10)$$

where $x_0 < \xi < x_2$.

Simpson's 3/8 rule with its error term is obtained using (2.5) with $n = 3,\ h = \frac{b-a}{3}, x_0 = a,\ x_2 = b,$ and (2.8) yields,

$$\int_a^b f(x)dx = \frac{3h}{8}\left(f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\right)+\frac{h^5 f^{(4)}(\xi)}{4!}\int_0^2 z\,(z-1)dz$$

$$= \frac{3h}{8}\left(f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\right)+\frac{h^5 f^{(4)}(\xi)}{4!}\left(-\frac{9}{10}\right)$$

$$\int_a^b f(x)dx = \frac{3h}{8}\left(f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\right)-\frac{3h^5 f^{(4)}(\xi)}{80} \qquad (2.11)$$

where $x_0 < \xi < x_2$.

These indicates that the degree of precision of (2.7) is $n+1$ and (2.8) is n, although the interpolation polynomial is of degree at most n.

2.4. Composite Rules

Composite quadrature rules for trapezoidal and Simpson's are simpler approach and is considered an effective way to increase the accuracy of the result when integrating high-degree polynomials. These rules involve partitioning the interval $[a, b]$ into subintervals then using the low-order interpolants such as the standard Newton-Cotes methods on each subinterval.

2.4.1. Composite Trapezoidal Rule

Suppose interval $[a, b]$ is partitioned into n intervals $[\,x_i,\ x_{i-1}]\,, i = 1, 2, \dots\dots n,$ by $a = x_0 < x_1 < \dots < x_k = b$. Employing the standard trapezoidal rule on n subinterval returns

$$\int_a^b f(x)dx = \sum_{i=1}^{n}\int_{x_i-1}^{x_i} f(x)dx \approx \sum_{i=1}^{n}\frac{x_i - x_{i-1}}{2}\left(f(x_{i-1})+f(x_i)\right). \qquad (2.12)$$

Since application of standard Newton-Cotes methods consider that f is evaluated at equally spaced nodes between a and b, $x_i = a + ih\ \ for\ i = 0, 1, 2, \dots\dots n\ and\ h = \frac{b-a}{n}$, then (2.12) returns the composite trapezoidal rule,

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left(f(a)+2\sum_{i=1}^{n-1} f(a+ih)+f(b)\right). \qquad (2.13)$$

Theorem2.3 [3, p. 206]Let $f \in C^2[a,b]$, $h = \frac{b-a}{n}$, and $x_i = a + ih \; for \; i = 0,1,2,\ldots..n$. There exists $\mu \in (a,b)$ for which the Composite Trapezoidal rule for n subintervals can be written with its error term as

$$\int_a^b f(x)\,dx = \frac{h}{2}\left(f(a) + 2\sum_{i=1}^{n-1} f(a+ih) + f(b)\right) - \frac{(b-a)\,h^2 f^{(2)}(\mu)}{12}. \tag{2.14}$$

The error term in (2.14) can be derived from the summation of all errors in the application of standard trapezoidal rule on each subinterval. Thus,

$$\begin{aligned}\int_a^b f(x)\,dx - \frac{h}{2}\left(f(a) + 2\sum_{i=1}^{n-1} f(a+ih) + f(b)\right) &= \sum_{i=1}^{n}\left(-\frac{(x_i - x_{i-1})^3 f^{(2)}(\xi_i)}{12}\right)\\ &= -\frac{h^3}{12}\sum_{i=1}^{n}\left(f^{(2)}(\xi_i)\right) \qquad \text{for } \xi_i \in [\,x_{i-1}, x_i\,].\end{aligned}$$

Since $f \in C^2[a,b]$ then by Extreme Value Theorem,

$$min_{x\,\in\,[a,b]} f^{(2)}(x) \le f^{(2)}(\xi_i) \le max_{x\,\in\,[a,b]} f^{(2)}(x)$$

we have, $\quad n\, min_{x\,\in\,[a,b]} f^{(2)}(x) \le \sum_{i=1}^{n}\left(f^{(2)}(\xi_i)\right) \le n\, max_{x\,\in\,[a,b]} f^{(2)}(x)$

and, $\quad min_{x\,\in\,[a,b]} f^{(2)}(x) \le \frac{1}{n}\sum_{i=1}^{n}\left(f^{(2)}(\xi_i)\right) \le max_{x\,\in\,[a,b]} f^{(2)}(x).$

By Intermediate Value Theorem, there is $\mu \in (a,b)$ such that

$$f^{(2)}(\mu) = \frac{1}{n}\sum_{i=1}^{n}\left(f^{(2)}(\xi_i)\right).$$

Hence, $-\frac{h^3}{12}\sum_{i=1}^{n}\left(f^{(2)}(\xi_i)\right) = -\frac{h^3\, n\, f^{(2)}(\mu)}{12}$,

and since $h = \frac{b-a}{n}$, then $\quad -\frac{h^3}{12}\sum_{i=1}^{n}\left(f^{(2)}(\xi_i)\right) = -\frac{(b-a)\,h^2 f^{(2)}(\mu)}{12}.$

It is intriguing to note that the error term for the composite trapezoidal rule is $\vartheta(h^2)$ not $\vartheta(h^3)$ that we have in (2.9). These are not comparable because for standard trapezoidal rule h is fixed at $h = b - a$ since $n = 1$, but for composite trapezoidal rule $h = \frac{b-a}{n}$, for positive integer n.

2.4.2. Composite Simpson's Rule

Composite Simpson's 1/3 rule is obtained in an equivalent manner (as Composite Trapezoidal rule); however, we choose an even integer n, partitioning the interval $[a,b]$ into n subintervals, then applying the standard Simpson's rule on each consecutive pair of subintervals. Applying some algebraic manipulations, we obtain

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\ \left(f(a) + 4\sum_{i=1}^{n/2} f(a+(2i-1)h) + 2\sum_{i=1}^{\frac{n}{2}-1} f(a+2ih) + f(b) \right). \tag{2.15}$$

Theorem 2.4 [3, p. 206] Let $f \in C^4[a,b]$, n be even, $h = \frac{b-a}{n}$, and $x_i = a + ih \;\; for\; i = 0, 1, \ldots \ldots n$. There exists $\mu \in (a,b)$ for which the composite Simpson's rule for n subintervals can be written with its error term as

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a) + 4\sum_{i=1}^{n/2} f(a+(2i-1)h) + 2\sum_{i=1}^{\frac{n}{2}-1} f(a+2ih) + f(b) \right] - \frac{(b-a)\,h^4 f^{(4)}(\mu)}{180}. \tag{2.16}$$

Similarly, the error term in (2.16) can be derived from the summation of all errors in the application of standard Simpson's rule on each consecutive pair of subintervals. Thus,

$$\int_a^b f(x)\,dx - \frac{h}{3}\left[f(a) + 4\sum_{i=1}^{\frac{n}{2}} f(a+(2i-1)h) + 2\sum_{i=1}^{\frac{n}{2}-1} f(a+2ih) + f(b) \right]$$

$$= \sum_{i=1}^{n/2}\left(-\frac{(x_{2i} - x_{2i-2})^5 f^{(4)}(\xi_i)}{2880} \right)$$

$$= -\frac{h^5}{90}\sum_{i=1}^{n/2} f^{(4)}(\xi_i),$$

for $\xi_i \in [\, x_{2i-2}, x_{2i}\,]$ for $i = 1, 2, \ldots \frac{n}{2}$.

Since $f \in C^4[a,b]$ then by Extreme Value Theorem,

$$min_{x \in [a,b]} f^{(4)}(x) \le f^{(4)}(\xi_i) \le max_{x \in [a,b]} f^{(4)}(x)$$

we have, $\frac{n}{2}\, min_{x \in [a,b]} f^{(4)}(x) \le \sum_{i=1}^{n/2}\left(f^{(4)}(\xi_i) \right) \le \frac{n}{2}\, max_{x \in [a,b]} f^{(4)}(x)$

and $min_{x \in [a,b]} f^{(4)}(x) \le \frac{2}{n}\sum_{i=1}^{\frac{n}{2}}\left(f^{(4)}(\xi_i) \right) \le max_{x \in [a,b]} f^{(4)}(x).$

By Intermediate Value Theorem, there is $\mu \in (a,b)$ such that,

$$f^{(4)}(\mu) = \frac{2}{n}\sum_{i=1}^{n/2}\left(f^{(4)}(\xi_i) \right).$$

Hence

$$-\frac{h^5}{90}\sum_{i=1}^{n/2} f^{(4)}(\xi_i) = -\frac{h^5\, n\, f^{(4)}(\mu)}{180},$$

and since $h = \frac{b-a}{n}$, then

$$-\frac{h^5}{90}\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\xi_i) = -\frac{(b-a)\,h^4\ f^{(4)}(\mu)}{180}.$$

We can also observe that the error term for the Composite Simpson's rule is $\vartheta(h^4)$, instead of $\vartheta(h^5)$ which was the error in the standard Simpson's rule (2.10). This is simply because the h in standard Simpson's rule is fixed at rule $h = \frac{b-a}{2}$ while in composite Simpson's rule it is rule $h = \frac{b-a}{n}$ for an even integer n.

2.5. Mathematica Software

MATHEMATICA is considered a definitive system for modern technical computing [18]. It is widely used in highly computational environments (both numerical and symbolic) because of its robust and comprehensive features, which allow flexibility and reliability. We use MATHEMATICA not only to ensure accuracy and precision of the results but also to visualize and observe the behaviours of the functions in the three methods employed.

3. Results and Discussion

This study investigates the robustness and accuracy of composite trapezoidal rule, Simpson's 1/3 rule, and Simpson's 3/8 rule to approximate the definite integrals of the standard normal probability density function, incomplete gamma function, the Fresnel sine function, and Dawson's integral These functions hold significant importance and find extensive applications in various disciplines such as Statistics, Business, and Engineering. The outcomes are discussed in this section.

3.1. Standard Normal Probability Density Function

The standard normal probability density function is given by $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. The function is the basis of calculating the area under the normal curve by determining the definite integral given a particular interval. However, due to the complexity of the function, it is tedious to determine the integral. For this reason, numerical integration is used to estimate the value within a given domain.

Table 1 displays the results of approximating standard normal probability density function from 0 to 1. This is particularly the area of the normal curve from $z - score = 0$ to $z - score = 1$. The table shows that both Simpson's 1/3 rule and 3/8 rules are the most efficient in terms approximating the integral since it only takes 2 subintervals, $n = 2$ and $n = 4$ for the 1/3 rule and $n = 3$ and $n = 6$ for the 3/8 rule for the approximate value to converge with actual value considering the absolute errors within 10^{-4} tolerance level. The trapezoidal rule on the other hand requires a significantly larger number of subintervals to achieve a comparable level of accuracy, requiring 21 subintervals to approach the exact value.

Table 1. Approximation of $\int_0^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ within 10^{-4}

n	Trapezoidal Rule			Simpson's 1/3 Rule			Simpson's 3/8 Rule		
	Approx. Answer	Exact Answer	Abs. Error	Approx. Answer	Exact Answer	Abs. Error	Approx. Answer	Exact Answer	Abs. Error
1	0.3204565025	0.3413447461	0.0209						
2	0.3362609146	0.3413447461	0.0051	0.3415290520	0.3413447461	0.0002			
3	0.3390959119	0.3413447461	0.0022				0.3414258380	0.3413447461	0.0001
4	0.3400818445	0.3413447461	0.0013	0.3413554879	0.3413447461	0.0000			
5	0.3405370985	0.3413447461	0.0008						
6	0.3407841090	0.3413447461	0.0006				0.3413495080	0.3413447461	0.0000
7	0.3409329509	0.3413447461	0.0004						
8	0.3410295157	0.3413447461	0.0003						
9	0.3410957025	0.3413447461	0.0002						
10	0.3411430365	0.3413447461	0.0002						
11	0.3411780536	0.3413447461	0.0002						
12	0.3412046843	0.3413447461	0.0001						
⋮	⋮	⋮	-do-						
21	0.3412990187	0.3413447461	0.0000						

Although both Simpson’s 1/3 and 3/8 rules require the same number of subintervals to achieve convergence, a closer examination reveals that the Simpson’s 3/8 rule yields a smaller absolute error before reaching convergence with the true value. It can be noted in Figure. 1 that there is a significant difference of the absolute error of the trapezoidal Rule compared to the other methods. A large gap of the errors is observed on the initial subintervals until the desired tolerance level is attained. On the other hand, there is a marginal difference between the errors of 1/3 rule and 3/8 rule.

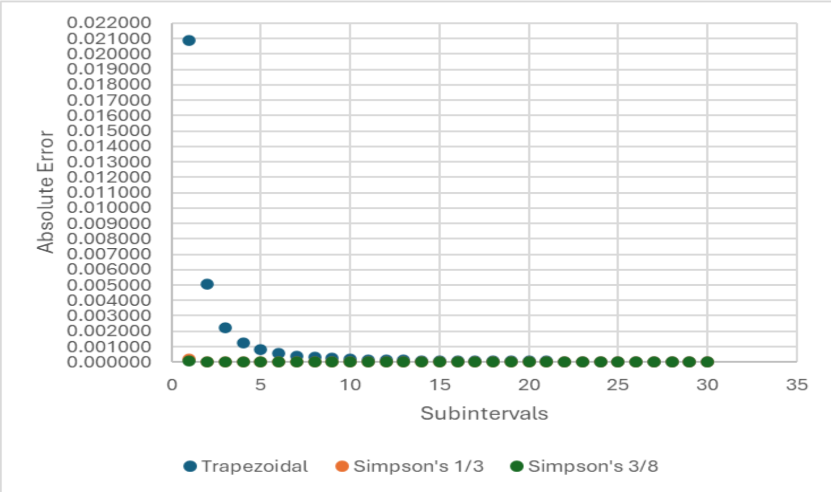

Figure 1. Absolute Error Graphs

Below are the Mathematica outputs of approximation of $\int_0^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ within $\mathbf{10^{-10}}$:

Absolute error set to be below:1.*10^-10
Searching for n that satisfies absolute error less than:1.*10^-10
**********************TRAPEZOIDAL RULE****************************
TOL met at n=9000, Absolute error at:2.489414785*10^-10
n=9000 approx: 0.3413447458 exact: 0.3413447461 Abs error : 2.489414785*10^-10
**********************SIMPSONS 1/3 RULE****************************
TOL met at n=74, Absolute error at:8.966438703*10^-11
n=74 approx: 0.3413447458 exact: 0.3413447461 Abs error : 8.966438703*10^-11
**********************SIMPSONS 3/8 RULE****************************
TOL met at n=90, Absolute error at:9.220813002*10^-11
n=90, approx: 0.3413447458 exact: 0.3413447461 Abs error : 9.220813002*10^-11

As revealed by the outputs, if we increase the tolerance level for absolute error to be within 10^{-10}, there is a significance increase of the values of the subintervals across the three methods for the approximate value to converge with the exact value. Specifically, it necessitates 9000 subintervals for the trapezoidal rule and 37 and 30 for 1/3 and 3/8 Simpsons rules, respectively. In the field of numerical integration, higher tolerance level indicates better precision and accuracy.

3.2. Fresnel Sine Function

The Fresnel sine integral, $f(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$ is widely used in the field of optics and used in the calculation of electromagnetic field density. The approximation of the function at $x = 1$ is scrutinized in this study. The results in Table 2 indicates that there is a lesser variation in terms of the convergence behaviour across the 3 methods in approximating the definite integral considering the tolerance level of 10^{-4}. It can be observed from the results that it only requires fewer subintervals of the intervals to achieve an approximation that closely aligns with the true value. The trapezoidal rule necessitated the use of 6 subintervals to produce an approximate value that converged with the exact value. In contrast, both Simpson's 1/3 and 3/8 rules only required 4 to reach a similar level of accuracy.

Table 2. Approximation of $\int_0^1 \sin\left(\frac{\pi t^2}{2}\right) dt$ within 10^{-4}

n	Trapezoidal Rule			Simpson's 1/3 Rule			Simpson's 3/8 Rule		
	Approx. Answer	Exact Answer	Error	Approx. Answer	Exact Answer	Error	Approx. Answer	Exact Answer	Error
1	0.5000000000	0.4382591474	0.0617						
2	0.4413417162	0.4382591474	0.0031	0.421788955	0.4382591474	0.0165			
3	0.4388119291	0.4382591474	0.0006				0.4311634203	0.4382591474	0.0071
4	0.4384277565	0.4382591474	0.0002	0.437456437	0.4382591474	0.0008			
5	0.4383270254	0.4382591474	0.0001						
6	0.4382915734	0.4382591474	0.0000	0.438118121	0.4382591474	0.0001	0.4379103055	0.4382591474	0.0003
8				0.438216495	0.4382591474	0.0000			
9							0.4381971694	0.4382591474	0.0001
12							0.4382403145	0.4382591474	0.0000

Additionally, the comparative analysis of the absolute errors across the three methods displayed in Figure 2 reveals that the Simpson's 3/8 rule consistently yields the minimal magnitudes of absolute errors until the desired precision is achieved. The first subinterval signifies larger disproportions on the absolute errors. The gaps are minimized when the succeeding subintervals are considered.

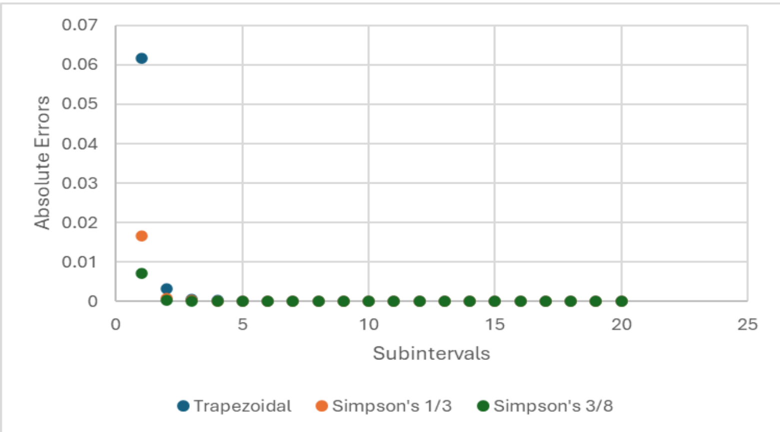

Figure 2. Absolute Error Graphs

Below are the Mathematica outputs of approximation of $\int_0^1 \sin\left(\frac{\pi t^2}{2}\right) dt$ within 10^{-10} when the tolerance level is increased to 10^{-10}. In contrast to the results using 10^{-4}, the marginal difference of the subintervals is maintained across the three methods. As per the table, the number of subintervals needed to maintain a comparable level of accuracy are: 142 for trapezoidal rule, 101 for Simpson’s 1/3 rule, and 83 for 3/8 rule.

Absolute error set to be below:1.*10^-10
Searching for n that satisfies absolute error less than:1.*10^-10
**********************TRAPEZOIDAL RULE****************************
TOL met at n=142, Absolute error at:9.834355552*10^-11
n=142 approx: 0.4382591475 exact: 0.4382591474 Abs error : 9.834355552*10^-1
**********************SIMPSONS 1/3 RULE****************************
TOL met at n=202, Absolute error at:9.880962715*10^-11
n=202 approx: 0.4382591475 exact: 0.4382591474 Abs error : 9.880962715*10^-11
**********************SIMPSONS 3/8 RULE****************************
TOL met at n=249, Absolute error at:9.6294972*10^-11
n=249, approx: 0.4382591475 exact: 0.4382591474 Abs error : 9.6294972*10^-11

3.3. Incomplete Gamma Function

The incomplete gamma function given by $\gamma(a,x) = \int_0^x t^{a-1}\, e^{-t} dt$ is commonly applied in the field of heat conduction, probability theory, and Fourier and Laplace transformations. The convergence analysis to approximate the value of the incomplete gamma function utilizing $a = 2$ on the interval $[0,1]$ within 10^{-4} is presented in Table 3.

Table 3. Approximation of $\int_0^1 \boldsymbol{t\, e^{-t} dt}$ within $\mathbf{10^{-4}}$

n	Trapezoidal Rule			Simpson's 1/3 Rule			Simpson's 3/8 Rule		
	Approx. Answer	Exact Answer	Error	Approx. Answer	Exact Answer	Error	Approx. Answer	Exact Answer	Error
1	0.1839397206	0.2642411177	0.0803						
2	0.2436025252	0.2642411177	0.0206	0.2634901268	0.2642411177	0.0008			
3	0.2550205234	0.2642411177	0.0092				0.2639056237	0.2642411177	0.0003
4	0.2590450402	0.2642411177	0.0052	0.2641925452	0.2642411177	0.0000			
5	0.2609128085	0.2642411177	0.0033						
6	0.2619287269	0.2642411177	0.0023				0.2642195021	0.2642411177	0.0000
7	0.2625417462	0.2642411177	0.0017						
8	0.2629398017	0.2642411177	0.0013						
9	0.2632127902	0.2642411177	0.0010						
10	0.2634080987	0.2642411177	0.0008						
n	Trapezoidal Rule								
	Approx. Answer	Exact Answer	Error						
11	0.2635526272	0.2642411177	0.0007						
12	0.2636625656	0.2642411177	0.0006						

13	0.2637481311	0.2642411177	0.0005		
14	0.2638160294	0.2642411177	0.0004		
15	0.2638708094	0.2642411177	0.0004		
16	0.2639156448	0.2642411177	0.0003		
17	0.2639528047	0.2642411177	0.0003		
18	0.2639839460	0.2642411177	0.0003		
19	0.2640103015	0.2642411177	0.0002		
: :	: :	: :	-do-		
24	0.2640964512	0.2642411177	0.0001		
: :	: :	: :	-do-		
41	0.2641915451	0.2642411177	0.0000		

There is an exceptionally large discrepancy in terms of the robustness of the trapezoidal rule as compared to Simpson's 1/8 and 3/8 rules. To arrive at the desired accuracy level, the trapezoidal rule required more than 20 times the number of subintervals of the Simpsons rules, $n = 2$ and $n = 4$ subintervals for the 1/3 rule and $n = 3$ and $n = 6$ for the 3/8 rule.

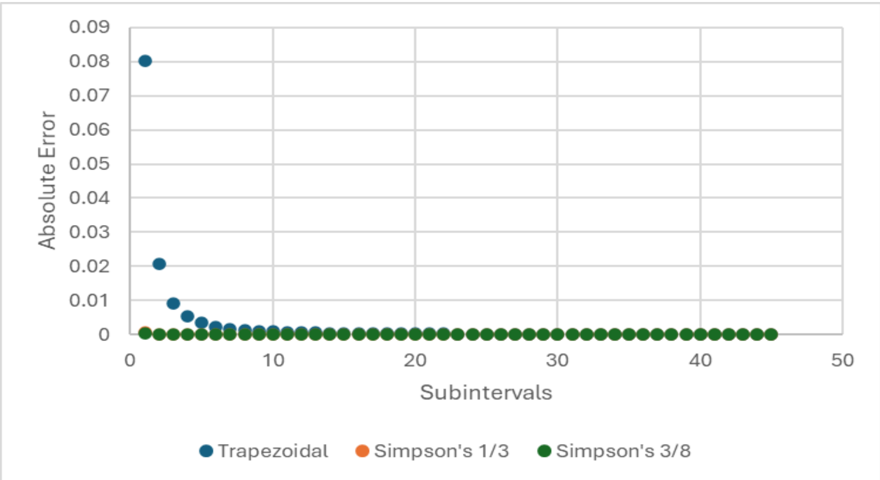

Figure 3. Absolute Error Graphs

Relative to the results of the other two functions, the 3/8 rule maintained the lowest absolute error as the value converges the exact answer as displayed in Figure 3. It can be noted from the graph that in the first few values of subintervals, the absolute errors of utilizing the trapezoidal method are significantly larger in contrast to the other 2 methods. Simpson's 1/3 and 3/8 rules exhibits no significant difference in the errors.

Mathematica outputs of approximation of $\int_0^1 t\,e^{-t}dt$ within $\mathbf{10^{-10}}$ below compares the trapezoidal rule, Simpson's 1/3 rule, and Simpson's 3/8 rule highlighting the number of subintervals needed to achieve an absolute error when the tolerance level is within 10^{-10}. The trapezoidal rule required 9000 subintervals to converge with the true value indicating slow convergence. This is a significant increase when compared to the 10^{-4} tolerance level. Simpson's 1/3 rule achieved the target accuracy with 52 subintervals while the Simpson's 3/8 rule met the tolerance with 44 subintervals.

Absolute error set to be below:1.*10^-10
Searching for n that satisfies absolute error less than:1.*10^-10
**********************TRAPEZOIDAL RULE*****************************
TOL met at n=9000, Absolute error at:1.028806873*10^-9
n=9000 approx: 0.2642411166 exact: 0.2642411177 Abs error : 1.028806873*10^-9
**********************SIMPSONS 1/3 RULE*****************************
TOL met at n=106, Absolute error at:9.963702086*10^-11
n=106 approx: 0.2642411166 exact: 0.2642411177 Abs error : 9.963702086*10^-11
**********************SIMPSONS 3/8 RULE*****************************
TOL met at n=132, Absolute error at:9.322431715*10^-11
n=132, approx: 0.2642411166 exact: 0.2642411177 Abs error : 9.322431715*10^-11

3.4. Dawson's Integral

The Dawson's integral, $f(x) = e^{-x^2}\int_0^1 e^{y2}dy$, bears important contribution in the concept of heat conduction and the theory of electrical oscillations of special vacuum tubes. Table 4 shows the results when approximating the value of the Dawson's integral on the interval $[0,1]$. Comparative analysis with the results obtained in incomplete gamma function shows the same convergence behaviour using the trapezoidal rule. The data further reveals that it takes 58 subintervals to attain the targeted level of precision. The magnitude is significantly larger as compared to the outcomes of Simpson's 1/3 and 3/8 rules. It is apparent on the results that the 3/8 rule is the most efficient method, requiring only 3 subintervals ($n = 3$, 6, and 9), in comparison to the 1/3 rule, which demands 4 subintervals ($n = 2$, 4, 6, and 8) for the approximate answer to converge with the exact answer.

Table 4. Approximation of $\int_0^1 \frac{1}{e} e^{t2}dt$ within $\mathbf{10^{-4}}$

n	Trapezoidal Rule			Simpson's 1/3 Rule			Simpson's 3/8 Rule		
	Approx. Answer	Exact Answer	Error	Approx. Answer	Exact Answer	Error	Approx. Answer	Exact Answer	Error
1	0.6839397206	0.5380795069	0.1459						
2	0.5781531367	0.5380795069	0.0401	0.5428909420	0.5380795069	0.0048			
3	0.5562684773	0.5380795069	0.0182				0.5403095719	0.5380795069	0.0022
4	0.5483901066	0.5380795069	0.0103	0.5384690966	0.5380795069	0.0004			
5	0.5447023760	0.5380795069	0.0066						
6	0.5426879211	0.5380795069	0.0046	0.5381610690	0.5380795069	0.0001	0.5382547691	0.5380795069	0.0002

7	0.5414693850	0.5380795069	0.0034						
8	0.5406769309	0.5380795069	0.0026	0.5381058724	0.5380795069	0.0000			
9	0.5401329056	0.5380795069	0.0021				0.5381159591	0.5380795069	0.0000
10	0.5397434061	0.5380795069	0.0017						
11	0.5394550259	0.5380795069	0.0014						
12	0.5392355782	0.5380795069	0.0012						
13	0.5390647298	0.5380795069	0.0010						
14	0.5389291253	0.5380795069	0.0008						
15	0.5388196999	0.5380795069	0.0007						
16	0.5387301253	0.5380795069	0.0007						
17	0.5386558760	0.5380795069	0.0006						
18	0.5385936459	0.5380795069	0.0005						
19	0.5385409745	0.5380795069	0.0005						
20	0.5384960001	0.5380795069	0.0004						
21	0.5384572932	0.5380795069	0.0004						
22	0.5384237410	0.5380795069	0.0003						
⋮	⋮	⋮	-do-						
26	0.5383259945	0.5380795069	0.0002						
⋮	⋮	⋮	-do-						
34	0.5382236615	0.5380795069	0.0001						
⋮	⋮	⋮	-do-						
58	0.5381290487	0.5380795069	0.0000						

As observed from Figure 4, the absolute errors of the trapezoidal rule considering the first few intervals exhibits a wide gap as compared to Simpson's 1/3 and 3/8 rules. Although there is a marginal difference on the errors between 1/3 and 3/8 rules, the latter bagged the lowest magnitude.

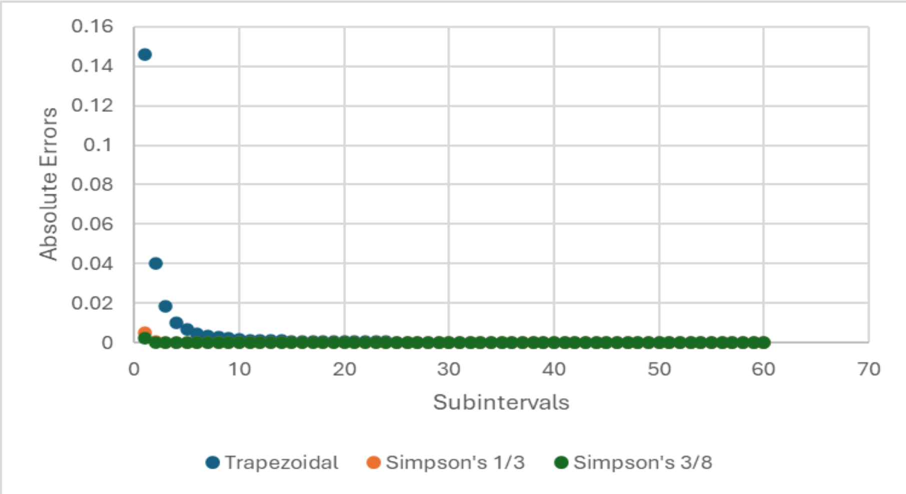

Figure 4. Absolute Error Graphs

When the tolerance level is increased to 10^{-10}, it can be observed from Table 8 that trapezoidal rule requires 9000 subintervals to achieve convergence. This is comparable to the outcomes of the incomplete gamma function and standard normal distribution function. Simpson's 3/8 Rule has the fastest convergence which necessitates the use of 75 subintervals while the 1/3 Rule requires 92 subintervals.

Absolute error set to be below:1.*10^-10
Searching for n that satisfies absolute error less than:1.*10^-10
**********************TRAPEZOIDAL RULE****************************
TOL at n=9000, Absolute error at:2.057612525*10^-9
n=9000 approx: 0.538079509 exact: 0.5380795069 Abs error : 2.057612525*10^-9
**********************SIMPSONS 1/3 RULE****************************
TOL met at n=184, Absolute error at:9.693024161*10^-11
n=184 approx: 0.538079509 exact: 0.5380795069 Abs error : 9.693024161*10^-11
**********************SIMPSONS 3/8 RULE****************************
TOL met at n=225, Absolute error at:9.753819974*10^-11
n=225, approx: 0.538079509 exact: 0.5380795069 Abs error: 9.753819974*10^-11

4. Conclusions

The results in the preceding section reveal distinct patterns regarding the efficiency and accuracy of various numerical integration methods across different classes of functions. Simpson's 1/3 and 3/8 rules consistently surpass the trapezoidal rule. The two methods show faster convergence and higher level of accuracy. However, when the tolerance level is increased to a considerable magnitude, the Simpson's 3/8 rule emerged as the most robust among the three methods. Moreover, comparative analysis of absolute errors reveals that the Simpson's 3/8 rule yields smaller magnitudes of absolute errors. Conversely, the trapezoidal rule consistently exhibits a slower convergence rate, which requires larger number of subintervals to attain the desired level of accuracy.

The four functions utilized a common interval $[0,1]$ for estimating the definite integral, thus it is recommended to investigate various domains to substantiate the findings and to gain a deeper understanding on the numerical methods utilized in the study. In addition to examining absolute error, future research should incorporate a comprehensive error analysis that includes relative error, truncation error, and round-off error and error bound. This can provide a more detailed understanding of the sources and magnitudes of errors in numerical integration. Furthermore, we recommend extending the research to include higher-dimensional integrals could provide valuable insights into the robustness of these numerical methods.

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