ON A DIOPHANTINE PROOFS OF FLT: THE FIRST CASE AND THE SECOND CASE $z \equiv 0 \ (modp)$ AND SIGNIFICANT RELATED PROBLEMS

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ABSTRACT

In this paper, we study Fermat's equation,

$$x^n + y^n = z^n \qquad (1)$$

with $n > 2, x, y, z$ non-zero positive integers. Let (a, b, c) be a triple of non-zero positive integers relativity prime. Consider the equation (1) with prime exponent $p > 2$. We establish the following results:

- $a^p + b^p \neq (b+1)^p$. *This completes the general direct proof of Abel's conjecture only prove in the first case $ab(b+1) \not\equiv 0 \ (mod\ p)$.*
- $a^{2p} + b^{2p} \neq c^{2p}$. *This completes the direct proof of Terjanian Theorem only prove in the first case $abc \not\equiv 0 \ (mod\ p)$).*
- $a^n + b^n \neq c^n$*withn is a non-prime integer.A new result almost absent in the literature of this problem.*
- *If $ab \not\equiv 0 \ (mod\ p)$ then$a^p + b^p \neq c^p$. This provides simultaneous Diophantine evidence for the first case oand the second case$c \equiv 0 \ (mod\ p)$ of FLT.*

We analyse each of the evidence from the previous results and propose a ranking in order of increasing difficulty to establish them.

KEYWORDS

Fermat Last Theorem (FLT), Fermat equation, Abel conjecture, the first case, the secund case, prime exponent, non-prime exponent, even exponent, the principal Kimou divisors.

1. INTRODUCTION.

In 1670 Fermat wrote that "It is impossible for a cube to be written as the sum of two cubes or for a fourth power to be written as the sum of two fourth powers or, in general, for any number equal to a power greater than two to be written as the sum of two powers" [1] p.1-2.Fermat claimed to have "woven" a wonderful proof of his problem. He gave the principle of is proof, the infinite descent, and illustrated it by proving the exponent 4 of his problem. For a little more than three centuries, Fermat's proposition, hitherto called Fermat's conjecture, had not yet been demonstrated in generality, even for the first case. However, non-obvious elementary proofs based on the principle of Fermat's infinite descent or not have been obtained for the small exponents of 3, 5, … ,100 (first case) and 3, …,14 (general case) [1] p. 64. Using computer tools, these limits had been pushed to $57*10^9$(Morishima and Gunderson, 1948) for the first case and to 125 000 (Wagstaff,1976) for the general case [1] p.19. Apart from these results concerning precise values of the exponents or its programming, there are other partial results involving families of prime exponents and based on relatively elementary theories [2] p. 109-122,203-211,360-361:

- In 1823, Sophie Germain and Legendre established the first case of FLT for exponents n less than 100. It also states that if $n = p$ is prime number such that $2p + 1$ is still prime, then the first case of FLT for exponent p is true.
- In 1846, Kummer used the theory of cyclotomic fields to obtain some very remarkable results: The impossibility of Fermat's equation for regular prime number n and deduce that the first case of FLT failsfor all prime exponents less than 100 except 37, 59 and 67.
- In 1977, Terjanian proved the first case of even exponent of FLT. He considered$x^{2p} + y^{2p} = z^{2p}$ with pa prime and he used the law of reciprocity to prove an important lemma involving quotients $\frac{z^p-y^p}{z-y}, \frac{z^q-y^q}{z-y}$ and Jacobi's symbols.

Despite these results, general proof was still slow to be found.It was in 1985 that Andrews Wiles provided the first recognized proof by the scientific community of Fermat's conjecture, which would become the Fermat-Wiles theorem [2] [3]. In 2023, Kimou K. P. took the decisive step by introducing Kimou 's divisors for a hypothetical solution of $x^n + y^n = z^n$ with $n = 4, p, 2p$and proposing new proofs of FLT for exponent 4, the first case of the Abel conjecture, and proved some properties related to Fermat problem [4]-[15]. Then, he proved new fundamental and decisive results for this problem: A crucial relationship and a fundamental theorem that will allow him to reach the "Heart" of the problem [10]-[11]. Then, in oral communication, he used them to prove the first and second cases $z \equiv 0\ (modp)$ of FLT [15].A solution (x, y, z) to the equation (1) will be called primitive if $\gcd(x, y, z) = 1$. This solution will be called trivial if $xyz = 0$. Let $n > 2$ a natural number. Consider the set $F_n$of triples of non-trivial positive integers solution to equation (1) define as follow:

$$F_n = \{(x, y, z) \in \mathbb{N}^{*3}, x^n + y^n = z^n\}.$$

The objective of this paper is to give a Diophantine proof for following main results.

Theorem 1.1. Let $p > 2$ be a primenumber and (a, b, c) be a triple of non-zero positive integers relatively prime. Then.

$$c - b = 1 \Longrightarrow a^p + b^p \neq c^p.$$

Theorem 1.2. Let pbe a prime number. Then,

$$p > 2 \Longrightarrow F_{2p} = \emptyset.$$

Theorem 1.3. Let nbe a nonprimepositive integer. Then,

$$n > 2 \Longrightarrow F_n = \emptyset.$$

Theorem 1.4. Let $p > 2$be a primenumber and let (a, b, c) be a triple of non-null positive integers relatively prime. Then

$$ab \not\equiv 0\ (mod\ p) \Longrightarrow a^p + b^p \neq c^p.$$

We present our work by organizing it as follows. Section 2 preliminaries, we recall the theorems of principal Kimou's divisors and Diophantine quotients and remainders, the Fundamental Relation of the Fermat equation and its corollary. In Section 3, we prove our main results. In section 4, we give a classification in increasing order of the difficulty of the Fermat problems studied here. In section 5, we conclude this work with a conclusion with perspectives.

2. PRELIMINARIES

In this section we define commonly used terms, state and prove theorems and lemmas necessary for the proofs of our main results.

Definitions 2.1.

1. Diophantine proof is direct proof based on the natural integers, using only the properties of addition, multiplication, Euclidean division, the order relation in $\mathbb{N}$ and the fundamental theorem of arithmetic to analyze a Diophantine equation.
2. A hypothetical solution $(a, b, c)$of Fermat's equation is primitive if $\gcd(a, b, c) = 1$.

Remark 2.1.

1. In our research on FLT, we use classical tools such as Newton's binomial formula, factorization, the fundamental theorem of arithmetic (implicitly), Fermat's little theorem and intensively modular arithmetic. We have developed some very effective new tools for analyzing the Fermat equation. These tools are all Diophantine [Definition 2.1].
2. If (a, b, c) is a non-trivial primitive solution of Fermat equation, then:

$$\gcd(a, b) = \gcd(a, c) = \gcd(b, c) = 1.$$

Notation 2.1.

1. We use the symbol ￿to represent the empty clause. It is the proposition that is always false or absurd.
2. Let $p > 2$ a prime number, $(a, b, c) \in F_p$ such that $a < b < c$. Let $T_p(x, y)$ be the quantity defined by
$$T_p(x, y) = \frac{y^p - x^p}{y - x} \text{with } x, y \in \{b, \pm a, c\}, y > x.$$
$T_p(x, y)$is a positive integer.

Theorem 2.1. (*Fermat's little theorem*). If p is a prime number, then for any integer a, where p does not divide a ($a \not\equiv 0 \ (mod\ p)$) the following holds

$$a^{p-1} \equiv 1 \ (mod\ p)$$

Proof. See [12] p.33

Theorem2.2. Let $n > 2$ be an odd integer and let $p > 2$be a prime number. Then

$$F_p = \emptyset \Longrightarrow F_n = \emptyset.$$

Proof. Proving Theorem 2.2. is equivalent to proving that if $F_n \neq \emptyset$ then $F_p \neq \emptyset$. We proceed by contraposed reasoning. Let us consider that $F_n \neq \emptyset$. We distinguish two cases.

On the one hand, if $n = 2^l$ with $l \geq 2$. Consider the case $l = 2$. In that case $n = 4$ and equation (1) becomes

$$x^4 + y^4 = z^4.$$

This is Fermat's biquadraticequation and we all know that it does not admit non-trivial solutions [2] p.13 (2C).

Consider the case where $l > 2$ then $n = 2^l \equiv 0 \ (mod\ 4)$. Therefore, there exists a natural number k such that $n = 4k$. Equation (1) becomes $x^{4k} + y^{4k} = z^{4k}$. As a result

$$x^{4k} + y^{4k} = z^{4k} \Longrightarrow \left(x^k\right)^4 + \left(y^k\right)^4 = \left(z^k\right)^4 \Longrightarrow ⍰.$$

Hence $n \neq 2^l$, with $l > 2$. In short $n \neq 2^l$ with $l \geq 2$.

On the other hand, if $n \neq 2^l$ then n admits a prime factor $q > 2$. There exist $k \geq 2$ such as $n = kq$. Then

$$\begin{aligned} &F_n \neq \emptyset \Longrightarrow \exists (a,b,c) \in F_n, abc \neq 1 \\ &\Longrightarrow (a,b,c) \in F_{kq} \Longrightarrow a^{kq} + b^{kq} = c^{kq} \\ &\Longrightarrow \left(a^k\right)^q + \left(b^k\right)^q = \left(c^k\right)^q \ with\ q > 2\ a\ prime \\ &\Longrightarrow \left(a^k, b^k, c^k\right) \in F_q, a^k b^k c^k \neq 0 \\ &\Longrightarrow F_q \neq \emptyset. \end{aligned}$$

Hence if $F_p = \emptyset$ then $F_n = \emptyset$.

Lemma 2.1. Let $p > 2$be a prime number and let $(a,b,c) \in F_p$ such that $b > a$. Consider (q_1, q_2) and (r_1, r_2) the quotients and the remainders of the Euclidean division of b and c by a: $b = aq_1 + r_1$ and $c = aq_2 + r_2$. Then,

$$b = a + 1 \Longrightarrow q_1 = q_2 = 1.$$

Proof. See [8].

Lemma 2.2. Let $p > 2$be a prime number and let $(a,b,c) \in F_p$ be a triple of primitive solution such that $b > a$. Consider $c = aq_2 + r_2\ with\ r_2 < a$and $e = \gcd(b, c-a)$. Then,

$$q_2 = 1 \Longrightarrow \begin{cases} r_2 = \dfrac{e^p}{p} \text{ if } b \equiv 0 \ (mod\ p) \\ r_2 = e^p \text{otherwise.} \end{cases}.$$

Proof. See [8].

Theorem 2.3. (*Kimou-Fermat***)**. Let $p > 2$be a prime number and let (a,b,c) be a triple of positive integers relativity prime such that $b > a$. Then,

$$(a,b,c) \in F_p \Longrightarrow \begin{cases} b - a = 1 \text{ if } c \equiv 1 \ (mod\ 2) \\ b - a = 2 \text{ otherwise.} \end{cases}.$$

Proof. See [10].

Lemma 2.3. Let $p > 2$be a prime number and let $(a,b,c) \in F_p$ be a triple of primitive solution such that $b > a$. Then

$$b - a = 1 \iff c \equiv 1 \ (mod\ 2).$$

Proof. According to the assumptions of the previous lemma, we have:

On the one hand, let us show that if $b - a = 1$ then $c \equiv 1 \ (mod\ 2)$. We have:

$$b - a = 1 \Longrightarrow b = a + 1$$
$$\Longrightarrow a, b \text{ are opposite parity}$$
$\Longrightarrow c$ is odd because $\gcd(a, b, c) = 1$.

Reciprocally, let us show that if $c \equiv 1 \ (mod\ 2)$ then $b - a = 1$. We proceeded by reasoning by absurd:

$$c \equiv 1 \ (mod\ 2)\ , b - a = 2 \Longrightarrow b, a \text{ have the same parity}$$
$$\Longrightarrow b, a \text{ are odd because } \gcd(a, b, c) = 1.$$
$$\Longrightarrow c \text{ is even}$$
$$\Longrightarrow ⍰.$$

Hence $b - a = 1$.

Lemma 2.4. Let $p > 2$ be a prime and let $(a, b, c) \in F_p$ be a triple of primitive solution such that $b > a$. Then

$$b - a = 2 \iff c \equiv 0 \ (mod\ 2).$$

Proof. Can be deduced by contraposition of the previous lemma.

Lemma 2.5. Let $p > 2$ be a prime number and let (a, b, c) be a triple of relativity primeintegers.

Then

$$(a, b, c) \in F_p \Longrightarrow \begin{cases} c - b = \dfrac{d^p}{\gcd(d, p)}, T_p(b, c) = \gcd(d, p)\alpha^p\ , a = d\alpha \\ c - a = \dfrac{e^p}{\gcd(e, p)}, T_p(a, c) = \gcd(e, p)\beta^p\ , b = e\beta \\ a + b = \dfrac{f^p}{\gcd(f, p)}, T_p(-a, b) = \gcd(f, p)\gamma^p\ , c = f\gamma. \end{cases}$$

where the sextuple $(d, e, f, \alpha, \beta, \gamma)$ of positive integers is the Kimou divisors of (a, b, c).

Proof. See [7], [9].

Remark 2.2.

1. The triple (d, e, f) of non-zero positive integers is called Kimou primaries divisors of (a, b, c) and defined by follow:
$$d = \gcd(a, c - b)\ , e = \gcd(b, c - a)\ and\ f = \gcd(c, a + b).$$
2. Lemma 2.5 is the concise and unified version of Lemmas 2.4, 2.5 and Remarks 2.3,2.4. in [8] pp. 87-89.
3. If $(a, b, c) \in F_p$ then

$$\begin{cases} \gcd(d,p)=\gcd(e,p)=\gcd(f,p)=1 \; if \; abc \not\equiv 0 \; (mod\, p) \\ \gcd(d,p)=p, \qquad \gcd(e,p)=\gcd(f,p)=1 \; if \; a \equiv 0 \; (mod\, p) \\ \gcd(e,p)=p, \gcd(d,p)=\gcd(f,p)=1 \; if \; b \equiv 0 \; (mod\, p) \\ \gcd(f,p)=p, \gcd(d,p)=\gcd(e,p)=1 \; if \; c \equiv 0 \; (mod\, p). \end{cases}$$

Lemma 2.6. Let $p > 2$be a prime number, let (a,b,c) be a triple of relativity prime integers. Then

$$(a,b,c) \in F_p \Longrightarrow \gcd(d,\alpha)=\gcd(e,\beta)=\gcd(f,\gamma)=1$$

where the sextuple $(d,e,f,\alpha,\beta,\gamma)$ of positive integers is the Kimou's divisors of (a,b,c) [Lemma 2.5].

Proof. Under the assumptions of the previous lemma, we have:

$$(a,b,c) \in F_p \Longrightarrow \begin{cases} d=\gcd(a,c-b) \\ e=\gcd(b,c-a) \; [Remark\; 2.2.][7]p.84 \\ f=\gcd(c,a+b) \end{cases}$$

$$\Longrightarrow \begin{cases} d=\gcd\left(a,\dfrac{d^p}{\gcd(d,p)}\right) \\ e=\gcd\left(b,\dfrac{e^p}{\gcd(e,p)}\right) \quad [Lemma\; 2.5] \\ f=\gcd\left(c,\dfrac{f^p}{\gcd(f,p)}\right) \end{cases}$$

$$\Longrightarrow \begin{cases} d=\gcd\left(d\alpha,\dfrac{d^p}{\gcd(d,p)}\right) \\ e=\gcd\left(e\beta,\dfrac{e^p}{\gcd(e,p)}\right) \quad [Lemma\; 2.5] \\ f=\gcd\left(f\gamma,\dfrac{f^p}{\gcd(f,p)}\right) \end{cases}$$

$$\Longrightarrow \begin{cases} 1=\gcd\left(\alpha,\dfrac{d^{p-1}}{\gcd(d,p)}\right) \\ 1=\gcd\left(\beta,\dfrac{e^{p-1}}{\gcd(e,p)}\right) \\ 1=\gcd\left(\gamma,\dfrac{f^{p-1}}{\gcd(f,p)}\right) \end{cases} \Longrightarrow \begin{cases} 1=\gcd(\alpha,d) \\ 1=\gcd(\beta,e) \\ 1=\gcd(\gamma,f). \end{cases}$$

Lemma 2.7. Let $p > 2$be a prime number and let $(a,b,c) \in F_p$ be a triple of primitive solution. Then

$$abc \not\equiv 0 \; (mod\, p) \Longrightarrow (c-b)(c-a)(a+b) \not\equiv 0 \; (mod\, p)$$

Proof. Let $p > 2$be a prime number and let $(a,b,c) \in F_p$ be a triple of non-zero primitive positive integers. Consider the sextuple $(d,e,f,\alpha,\beta,\gamma)$of positive integers, its Kimou divisors. We have

$$abc \not\equiv 0 \; (mod\, p) \Longrightarrow def \not\equiv 0 \; (mod\, p) \text{otherwise } abc \equiv 0 \; (mod\, p)$$

$$\Rightarrow d \not\equiv 0\ (mod\ p), e \not\equiv 0\ (mod\ p), f \not\equiv 0\ (mod\ p)$$
$$\Rightarrow d^p \not\equiv 0\ (mod\ p), e^p \not\equiv 0\ (mod\ p), f^p \not\equiv 0\ (mod\ p)$$
$$\Rightarrow c - b \not\equiv 0\ (mod\ p), c - a \not\equiv 0\ (mod\ p), a + b \not\equiv 0\ (mod\ p)\ [Lemma\ 2.5.]$$

Lemma 2.8. Let $p > 2$be a prime number and $(a, b, c) \in F_p$ be a triple of primitive solution. Consider the triple (α, β, γ) theauxiliary Kimou divisors of (a, b, c).Then

$$\alpha \equiv \beta \equiv \gamma \equiv 1\ (mod\ p)$$

Proof. Let's deal with the first case of this problem. We have

$$abc \not\equiv 0\ (mod\ p) \Rightarrow T_p(a, c) = \frac{c^p - a^p}{c - a}$$
$$\Rightarrow T_p(a, c) \equiv \frac{c^p - a^p}{c - a} (mod\ p) [Lemma\ 2.7.]$$
$$\Rightarrow T_p(a, c) \equiv \frac{c - a}{c - a} (mod\ p)\ \ [Theorem\ 2.1.]$$
$$\Rightarrow T_p(a, c) \equiv 1\ (mod\ p)$$
$$\Rightarrow \alpha^p \equiv 1\ (mod\ p)\ [Lemma\ 2.5]$$
$$\Rightarrow \alpha \equiv 1\ (mod\ p)\ \ [Theorem\ 2.1.]$$

The same approach is followed to show that $\beta \equiv \gamma \equiv 1\ (mod\ p)$.

In the second case, let us illustrate the evidence on the case $a \equiv 0\ (mod\ p)$. On the one hand,

$$a \equiv 0\ (mod\ p) \Rightarrow a^p + b^p = c^p$$
$$\Rightarrow b^p \equiv c^p\ (mod\ p^p) \Rightarrow b \equiv c\ (mod\ p^{p-1})$$
$$\Rightarrow b \equiv c\ (mod\ p^2)\ because\ p \geq 3)$$

On the other hand,

$$a \equiv 0\ (mod\ p) \Rightarrow T_p(b, c) = p\alpha^p\ [Lemma\ 2.5, Remark\ 2.2.]$$
$$\Rightarrow T_p(b, c) \equiv p\alpha^p\ (mod\ p^2)$$
$$\Rightarrow pb^{p-1} \equiv p\alpha^p (mod\ p^2), using\ the\ previous\ result$$
$$\Rightarrow b^{p-1} \equiv \alpha^p\ (mod\ p)$$
$$\Rightarrow 1 \equiv \alpha^p (mod\ p)\ \ [Theorem\ 2.1]$$

$\Rightarrow 1 \equiv \alpha\ (mod\ p)$ $\quad [Theorem\ 2.1]$.

A similar approach is followed to deal with cases $bc \equiv 0\ (mod\ p)$.

Remark 2.3. Let $p > 2$be a prime number and let (a, b, c)be a triple of relativity prime integers**.** Then

$$(a, b, c) \in F_p \Rightarrow \alpha \geq p, \beta \geq p, \gamma \geq p \Rightarrow \alpha > 2, \beta > 2, \gamma > 2.$$

3. PROOF OF OUR MAIN RESULTS

3.1. Proof of Theorem 1.1.

Conjecture (Abel). Let $p > 2$be a prime number and let $(a, b, c) \in F_p$ be a triple of primitive solution. Then none of the $a, b$and$c$ is the power of a prime number.

The cases where b and c are powers of a prime number have been proved by Moller [14]. He also proved that if a is a prime power, then $c - b = 1$ [13], [14].The first case of this conjecture was proved by Abel himself. New evidence was given by Kimou P. in 2023 [6]. The second case has yet to receive direct proof. That's precisely the aim of this subsection.In what follows, we prove this conjecture in full.

Lemma 3.1. Let $p > 2$be a prime and let $(a, b, c) \in F_p$be a triple of primitive solution such that $abc \equiv 0 \ (mod\ p)$. If $\forall x \in \{a, b, c\}, \nexists(\pi, m) \in \mathbb{N}^{*2}$with $\pi$is a prime number and $x = \pi^m$then

$$x \not\equiv 0 \ (mod\ p).$$

Proof. Under the assumptions of the previous lemma, we will proceed by absurdity, assuming that $x \equiv 0 \ (mod\ p)$.

On the one hand,

$$\begin{gathered} b = \pi^m, b \equiv 0 \ (mod\ p) \Longrightarrow \pi^m \equiv 0 \ (mod\ p) \\ \Longrightarrow \pi \equiv 0 \ (mod\ p) \\ \Longrightarrow \pi = p \ or\ p = 1 \Longrightarrow \pi = p. \end{gathered}$$

On the other hand, according to the above, we have:

$$\begin{gathered} b = \pi^m, b \equiv 0 \ (mod\ p) \Longrightarrow e\beta = p^m \qquad [Lemma\ 2.5] \\ \Longrightarrow e\beta \equiv 0 \ (mod\ p) \\ \Longrightarrow e \equiv 0 \ (mod\ p) \ \ [Lemma\ 2.8]) \\ \Longrightarrow e \equiv 0 \ (mod\ p^m) \\ \Longrightarrow e = kb = b \ with\ k \geq 1 \ is\ a\ integer \\ \Longrightarrow e = b, \beta = 1 \\ \Longrightarrow \square \ [Remark\ 2.3]. \end{gathered}$$

Hence$b \not\equiv 0 \ (mod\ p)$.We proceed in the same way with a and c.

Remark 3.1. When b is even, it is treated as follows.If b is even, then $b = 2^m$ and consequently $b \not\equiv 0 \ (modp)$. According to lemma 3.1. we treat in the following the triplets $(a, b, c) \in F_p$ such that

$bc \not\equiv 0 \ (modp)$ with $c = b + 1$.

Lemma 3.2. Let $p > 2$ be a prime and let $(a, b, c) \in F_p$ be a triple of primitive solution such that $e = \gcd(b, c - a)$ and $b = e\beta$. Then

$$\beta > e^{p-1}.$$

Proof.

$$\begin{gathered} (a, b, c) \in F \Longrightarrow a + b - c = b - (c - a) = e\beta - e^p \\ \Longrightarrow a + b - c = e(\beta - e^{p-1}) \\ \Longrightarrow \beta - e^{p-1} > 0 \Longrightarrow \beta > e^{p-1}. \end{gathered}$$

Lemma 3.3. Let $p > 2$ be a prime and let $(a, b, c) \in F_p$ be a triple of primitive solution such that $e = \gcd(b, c - a)$ and $b = e\beta$. Then

$$\begin{cases} \gamma > \dfrac{f^{p-1}}{2p} \; if \; c \equiv 0 \; (mod\, p) \\ \gamma > \dfrac{f^{p-1}}{2p} \; otherwise \end{cases}.$$

Proof. Consider the assumptions of the previous lemma and $(a, b, c) \in F_p$ a triple of primitive solution. On the one hand

$$c \equiv 0 \; (mod\, p) \Longrightarrow 2c = d^p + e^p + \frac{f^p}{p} \; [7]$$
$$\Longrightarrow 2c > \frac{f^p}{p}$$
$$\Longrightarrow 2f\gamma > \frac{f^p}{p} \Longrightarrow \gamma > \frac{f^{p-1}}{2p}.$$

On the other hand

$$c \not\equiv 0 \; (mod\, p) \Longrightarrow 2c = \frac{d^p}{p} + e^p + f^p or \; 2c = d^p + \frac{e^p}{p} + f^p \; [7]$$
$$\Longrightarrow 2c > f^p$$
$$\Longrightarrow 2f\gamma > f^p \Longrightarrow \gamma > \frac{f^{p-1}}{2}.$$

Remark. If $c \equiv 0 \; (mod\, p)$ then $c \equiv 0 \; (mod\, p^2)$[2] p. Hence $f \equiv 0 \; (modp^2)$as a result: $f \geq p^2 > 4$. When$c \not\equiv 0 \; (modp)$let consider the following case:

$$b \equiv 0 \; (mod\, p) \Longrightarrow c \equiv a \; (mod\, p^2) \Longrightarrow f \equiv d \; (mod\, p^2)$$
$$\Longrightarrow f = d + kp^2 \; \Longrightarrow f > p^2$$

The same procedure is followed for the other cases.

Lemma 3.4. Let $p > 2$be a prime and let $(a, b, c) \in F_p$. Consider $e = \gcd(b, c - a)$. Then

$$b \not\equiv 0 \; (mod\, p) \Longrightarrow \nexists(\pi, m) \in \mathbb{N}^{*2}, b = \pi^m$$

with π is prime number.

Proof. Let $b \not\equiv 0 \; (mod\, p)$. We proceed by reasoning by the absurd. Let's assume that $(a, b, c) \in F_p \; andb = \pi^m$.We have:

$$b = \pi^m \Longrightarrow e\beta = \pi^m \;\; [Lemma\; 2.5]$$
$$\Longrightarrow e = 1 \; car \; \beta > e \; [Lemma\; 3.2]$$
$$\Longrightarrow e^p = 1 \Longrightarrow c - a = 1 \; [Lemma\; 2.5]$$
$$\Longrightarrow c = a + 1$$
$$\Longrightarrow ⍰ \text{ because } a < b < c.$$

Hence $\nexists(\pi, m) \in \mathbb{N}^{*2}, b = \pi^m$ with π a prime number.

Lemma3.5. Let $p > 2$be a prime and let $(a, b, c) \in F_p$ be a triple of primitive solution. Then

$$c \not\equiv 0 \; (mod\, p) \Longrightarrow \nexists(\pi, m) \in \mathbb{N}^{*2}, c = \pi^m \text{with } \pi \text{ is a prime}$$

Proof. Let $(a,b,c) \in F_p$ and $c \not\equiv 0 \ (mod\ p)$. We reason from the absurd by supposing that $c = \pi^m$.We have

$$(a,b,c) \in F_p, c \not\equiv 0 \ (mod\ p) \Longrightarrow f\gamma = \pi^m$$
$$\Longrightarrow f = 1\ because\ \gamma > f\ [Lemma\ 3.3.]$$
$$\Longrightarrow f^p = 1$$
$$\Longrightarrow a + b = 1\ [Lemma\ 2.5]$$
$$\Longrightarrow ab = 0 \Longrightarrow ⍰.$$

Hence $\nexists(\pi, m) \in \mathbb{N}^{*2}, c = \pi^m$ with π a prime number.

Lemma 3.6. Let $\pi > 2$be a prime and let $(a,b,c) \in F_p$be a triple of primitive positive integers solution of equation (1).Then,

$$a = \pi^m \Longrightarrow c - b = 1.$$

Proof. Under the assumptions of lemma 3.4. we proceedby absurd supposing that $c - b > 1$. We have:

$$c - b > 1 \Longrightarrow d\alpha = \pi^m, d^p > 1\ [Lemmas\ 2.5, 2.6]$$
$$\Longrightarrow d\alpha = \pi^m, d > 1$$
$$\Longrightarrow d\alpha = \pi^m, \quad d > 1, \alpha > d > 1.$$
$$\Longrightarrow ⍰ \text{ because } \gcd(d,\alpha) = 1\ [Lemma\ 2.6].$$

Hence $c - b = 1$.

Lemme 3.7. Let $p > 2$be a prime number and let $(a,b,c) \in F_p$be a triple of primitive solution. Then

$$c - b = 1 \Longrightarrow a \equiv 1\ (mod\ 2).$$

Proof.

$$c - b = 1 \Longrightarrow c = b + 1$$
$$\Longrightarrow c \text{ or } b \text{ is even}$$
$$\Longrightarrow a \text{ is odd.}$$

Lemme 3.8. Let $p > 2$ be a prime number and let $(a,b,c) \in F_p$ be a triple of primitive solution. Then

$$c - b = 1 \Longrightarrow \begin{cases} c - a = 2 \text{ if } c \equiv 1\ (mod\ 2) \\ c - a = 3 \text{ otherwise} \end{cases}$$

Proof. Under the assumptions of the previous lemma, we have:

$$c - b = 1 \Longrightarrow b = c - 1$$
$$\Longrightarrow \begin{cases} b - a = 1 \text{ if } c \equiv 1\ (mod\ 2) \\ b - a = 2 \text{ otherwise} \end{cases} [Theorem\ 2.3.]$$
$$\Longrightarrow \begin{cases} c - 1 - a = 1 \text{ if } c \equiv 1\ (mod\ 2) \\ c - 1 - a = 2 \text{ otherwise} \end{cases}$$
$$\Longrightarrow \begin{cases} c - a = 2 \text{ if } c \equiv 1\ (mod\ 2) \\ c - a = 3 \text{ otherwise} \end{cases}$$

Lemma 3.9. Let $p > 2$ be a prime number and let (a, b, c) a triple of non-null positive integers relativity primesuch that $c - b = 1$. Then

$$b \not\equiv 0\ (mod\ p) \Longrightarrow a^p + b^p \neq c^p$$

Proof. Under the assumptions of the previous lemma, we have if $b \not\equiv 0\ (mod\ p)$. Consider v_1 and v_2 respectively the 2-adic and 3-adic valuations of e. Then

$$c - b = 1 \Longrightarrow \begin{cases} c - a = 2 \text{ if } c \equiv 1\ (mod\ 2) \\ c - a = 3 \text{ otherwise} \end{cases} \quad [Lemma\ 3.8]$$

$$\Longrightarrow \begin{cases} e^p = 2\ if\ c \equiv 1\ (mod\ 2) \\ e^p = 3 \text{ otherwise} \end{cases} \quad [Lemma\ 2.5, Remark\ 2.2]$$

$$\Longrightarrow \begin{cases} k_1^p 2^{v_1 p - 1} = 1 \text{ if } c \equiv 1\ (mod\ 2) \\ k_2^p 3^{v_2 p - 1} = 1 \text{ otherwise} \end{cases} \text{with } e = k_1 2^{v_1} = k_2 3^{v_2}$$

$$\Longrightarrow \begin{cases} 2^{v_1 p - 1} = 1 \text{ if } c \equiv 1\ (mod\ 2) \\ 3^{v_2 p - 1} = 1 \text{ otherwise} \end{cases}$$

$$\Longrightarrow \begin{cases} p = \frac{1}{v_1} \text{if } c \equiv 1\ (mod\ 2) \\ p = \frac{1}{v_2} \text{otherwise} \end{cases} \Longrightarrow \begin{cases} p = 1 \text{ if } c \equiv 1\ (mod\ 2) \\ p = 1 \text{ otherwise} \end{cases}$$

$$\Longrightarrow p = 1 \Longrightarrow ⍰ \text{ because } p > 2.$$

Hence $c - b > 1$.

Remark 3.2. Let $p > 2$ be a prime number and let (a, b, c) a triple of non-null positive integers relativity prime such that $c - b = 1$. When $c \equiv 0\ (modp)$, we have $b \not\equiv 0\ (modp)$ and consequently, according to the preceding Lemma, $a^p + b^p \neq c^p$.

Lemma 3.10. Let $p > 2$ be a prime number and let (a, b, c) a triple of non-null positive integers relativity prime such that $c - b = 1$. Then

$$b \equiv 0\ (mod\ p) \Longrightarrow a^p + b^p \neq c^p$$

Proof. Under the assumptions of the previous lemma, we have$\equiv 0\ (mod\ p)$. Consider v_1, $v_2$and v_3 respectively the 2-adic, 3-adic and p-adic valuations of e. Then

$$b \equiv 0\ (mod\ p) \Longrightarrow \begin{cases} c - a = 2 \text{ if } c \equiv 1\ (mod\ 2) \\ c - a = 3 \text{ otherwise} \end{cases} \quad [Lemma\ 3.8]$$

$$\Longrightarrow \begin{cases} \frac{e^p}{p} = 2 \text{ if } c \equiv 1\ (mod\ 2) \\ \frac{e^p}{p} = 3 \text{ otherwise} \end{cases} \quad [Lemma\ 2.5, Remark\ 2.2]$$

$$\Longrightarrow \begin{cases} k_1^p 2^{v_1 p - 1} = p \text{ if } c \equiv 1\ (mod\ 2) \\ k_2^p 3^{v_2 p - 1} = p \text{ otherwise} \end{cases}$$

$$\Longrightarrow \begin{cases} k_1^p 2^{v_1 p - 1} p^{v_3 p - 1} = 1 \text{ if } c \equiv 1\ (mod\ 2) \\ k_2^p 3^{v_2 p - 1} p^{v_3 p - 1} = 1 \text{ otherwise} \end{cases}$$

$$\Longrightarrow \begin{cases} k_1 = k_2 = 1 \\ 2^{v_1 p - 1} p^{v_3 p - 1} = 1 \text{ if } c \equiv 1\ (mod\ 2) \\ 3^{v_2 p - 1} p^{v_3 p - 1} = 1 \text{ otherwise} \end{cases}$$

$$\Rightarrow \begin{cases} v_1 p - 1 = 0, \quad v_3 p - 1 = 0 \text{ if } c \equiv 1\ (mod\ 2) \\ v_2 p - 1 = 0, v_3 p - 1 = 0\ otherwise \end{cases}$$

$$\Rightarrow \begin{cases} p = \dfrac{1}{v_1}, \quad p = \dfrac{1}{v_3} \text{if } c \equiv 1\ (mod\ 2) \\ p = \dfrac{1}{v_2}, \quad p = \dfrac{1}{v_3},\ otherwise \end{cases} \Rightarrow \begin{cases} v_1 = v_2 = v_3 = 1 \\ p = 1 \end{cases}$$

$$\Rightarrow p = 1 \Rightarrow ⍰.$$

Hence $c - b > 1$.

Remark 3.3. Let $p > 2$ be a prime number and let (a, b, c) a triple of non-null positive integers relativity prime such that $c - b = 1$. When $c \not\equiv 0 (modp)$, we have $b \equiv 0\ (modp)$or $b \not\equiv 0\ (modp)$. In both cases, lemmas 3.9 and 3.10 confirm that $a^p + b^p \neq c^p$.

Proof of Theorem 1.1. Immediate consequences of Lemmas 3.9 and 3.10, and Remark 3.2 and 3.3.

3.2. Proof of Theorem 1.2

Proof. Let $p > 2$ be a prime number and let (a, b, c) be a triple of relativity prime integers. Then

$$(a, b, c) \in F_{2p} \Rightarrow (a^2, b^2, c^2) \in F_p, c \equiv 1\ (mod\ 2)$$

$\Rightarrow b^2 - a^2 = 1$ [Theorem 2.3.]

$$\Rightarrow (b - a)(a + b) = 1$$
$$\Rightarrow a + b = 1\ et\ b - a = 1$$

$\Rightarrow b = 1 \Rightarrow$⍰ because $b > 1$.

Hence the result.

Remark 3.4. Because ofTheorem1.2 and [1] p.13 (2C), Fermat's theorem is true for all even exponents.

3.3. Proof of Theorem 1.3

3.3.1. Proof of FLT for Odd No-Prime Exponent

Theorem 3.3. Let $m > 2$be a positive integer. Then.

$$m\ is\ an \text{ odd nonprime integer} \Rightarrow F_m = \emptyset.$$

Proof. Let $m > 2$be an odd no-prime number. $(a, b, c) \in F_m$. Then:

$$m \text{ is odd nonprime integer} \Rightarrow \exists (s, k), s > 2, k > 2 \text{ are odd prime}, a^{ks} + b^{ks} = c^{ks}$$
$$\Rightarrow (a^k)^s + (b^k)^s = (c^k)^s$$
$$\Rightarrow b^k - a^k = 1 \text{ or} b^k - a^k = 2\ \ [Theorem\ 2.3]$$
$$\Rightarrow (b - a)T_k(a, b) = 1 \text{ or } b - a = 2 \text{ because } k \text{ is odd}$$
$$\Rightarrow 1 > k(b - a)a^{k-1} \text{or } 2 > k(b - a)a^{k-1}$$
$$\Rightarrow b = a \Rightarrow ⍰;$$

Hence m cannot be an odd nonprime integer and consequently $F_m = \emptyset$.

Remark 3.5. FLT is true for odd nonprime exponent.

3.3.2. Proof FLT for Nonprime Exponent

Under the assumptions of Theorem 1.3. let consider$(a, b, c) \in F_n$ be a triple of primitive integers with na nonprime positive integer. We proceed by the absurd:

$$n \equiv 0\ (mod 2) \Longrightarrow \exists (l, q) \in \mathbb{N}^2, l > 1, q > 3\ anoddinteger, n = 2q or 2^l q$$
$$\Longrightarrow a^{2q} + b^{2p} = c^{2q}\ or\ a^{2^l q} + b^{2^l p} = c^{2^l q}$$
$$\Longrightarrow \exists\ p > 3\ a\ prime, l_1 \geq 1, a^{2pq_1} + b^{2pq_1} = c^{2pq_1}\ or\ a^{4l_1 q} + b^{4l_1} = c^{4l_1 q}$$
$$\Longrightarrow\ (a^{q_1})^{2p} + (a^{q_1})^{2p} = (a^{q_1})^{2p}\ or\ \left(a^{l_{1q}}\right)^4 + \left(a^{l_1 q}\right)^4 = \left(c^{l_1 q}\right)^4$$

⟹⍰. [Theorem 1.2] [2] p.13 (2C).

Hence $n \not\equiv 0\ (mod 2)$. Par suite $n \equiv 1\ (mod 2)$. Traitons ce cas :

$$n \equiv 1\ (mod 2) \Longrightarrow n\ is\ an\ odd\ nonprime\ integer$$
$$\Longrightarrow F_n = \emptyset.$$

Hence if n is nonprime integer FLT is true. This proves Theorem 1.3.

3.4. Proof of Theorem 1.4

In this section we prove the first case of FLT and the second case $z \equiv 0\ (mod\ p)$. We distinguish two new cases: The case $c \equiv 1\ (mod\ 2)$ or $c \equiv 0\ (mod\ 2)$

Lemma 3.8. Let $p > 2$ be a prime number and let (a, b, c) be a triple of relativity prime integers. Then

$$(a, b, c) \in F_p \Longrightarrow \begin{cases} b - a = e^p - d^p\ if\ ab \not\equiv 0\ (mod\ p) \\ b - a = e^p - \dfrac{d^p}{p}\ if\ a \equiv 0\ (mod\ p) \\ b - a = \dfrac{e^p}{p} - d^p\ if\ b \equiv 0\ (mod\ p) \end{cases}$$

where (d, e) is the couple of Kimou's primaries divisors of (a, b).

Proof. According to [7], [9], we have

If$(a, b, c) \in F_p$ then

$$b - a = \left(-\frac{d^p}{\gcd(d, p)} + \frac{e^p}{\gcd(e, p)} + \frac{f^p}{\gcd(f, p)}\right) - \left(\frac{d^p}{\gcd(d, p)} - \frac{e^p}{\gcd(e, p)} + \frac{f^p}{\gcd(f, p)}\right)$$
$$= \frac{e^p}{\gcd(e, p)} - \frac{d^p}{\gcd(d, p)}.$$

Hence,

$$(a,b,c) \in F_p \Longrightarrow \begin{cases} b-a = e^p - d^p \; if \; ab \not\equiv 0 \; (mod \; p) \\ b-a = e^p - \dfrac{d^p}{p} \; if \; a \equiv 0 \; (mod \; p) \\ b-a = \dfrac{e^p}{p} - d^p \; if \; b \equiv 0 \; (mod \; p) \end{cases} \; [Remark \; 2.2.]$$

Proof of Theorem 1.4. Under the assumptions of the Theorem 1.4. we proceed to a proof by the absurd by assuming that $(a,b,c) \in F_p$.

We have:

$$ab \not\equiv 0 \; (mod \; p) \Longrightarrow \begin{cases} b-a = 1 \;\; if \; c \equiv 1 \; (mod \; 2) \\ \quad b-a = 2 \; otherwise \end{cases} \quad [Theorem \; 2.2.]$$
$$\Longrightarrow \begin{cases} e^p - \; d^p = 1 \;\; if \; c \equiv 1 \; (mod \; 2) \\ \quad e^p - \; d^p = 2 \; otherwise \end{cases} = 1 \; [Lemme \; 3.8]$$
$$\Longrightarrow \begin{cases} 1 > p(e-d) \; d^{p-1} if \; c \equiv 1 \; (mod \; 2) \\ \quad 2 > p(e-d) \; d^{p-1} \; otherwise \end{cases}$$
$$\Longrightarrow 2 > p(e-d) \; d^{p-1}$$
$$\Longrightarrow (e-d) \; d^{p-1} < \frac{2}{p} < 1$$
$$\Longrightarrow e = d \Longrightarrow ⍰.$$

Hence the result.

Remark 3.6. We have just proved Fermat's last theorem with the even exponent, in its first case and in the second case where $c \equiv 0 \; (mod \; p)$. However, when $ab \equiv 0 \; (mod \; p)$ we have:

$$a \equiv 0 \; (mod \; p) \Longrightarrow \begin{cases} e^p - \dfrac{d^p}{p} = 1 \; if \; c \equiv 1 \; (mod \; 2) \\ \quad e^p - \dfrac{d^p}{p} = 2 \;\; \text{otherwise} \end{cases},$$

and

$$b \equiv 0 \; (mod \; p) \Longrightarrow \begin{cases} \dfrac{e^p}{p} - d^p = 1 \; if \; c \equiv 1 \; (mod \; 2) \\ \quad \dfrac{e^p}{p} - d^p = 2 \;\; \text{otherwise.} \end{cases}$$

These new Diophantine equations promise to be difficult despite their simple appearances.

4. ANALYSIS OF THE DIFFICULTY OF ESTABLISHING RESULTS

At this stage we propose a classification by increasing difficulty of solving the problems dealt with in this paper. First place is occupied by the second FLT case with the odd exponent: it is obvious that if$(a,b,c) \in F_{2p}$, with pa prime number, then $b^2 - a^2 = 1$is impossible by simple making factorisation. The second position is occupied simultaneously by the first and second FLT cases $c \equiv 0 \; (mod \; p)$: if $(a,b,c) \in F_p$Then $e^p - d^p = 1$. Then you'll have to factor and major. You'll conclude that this relationship is impossible. This case is more difficult than the previous one. The third place is occupied by FLT with nonprime exponent. To prove this, we had to distinguish two sub-problems: Prove that FLT is true for the odd non-prime exponent and then for the even exponent.In the last position is the second case of Abel's conjecture. Indeed, it turned out to be a little more difficult than previous problem because it was necessary to use the relations $b - a = 1,\ c - b = 1$, the Kimou's principal divisors and p adic valuations to establish a contradiction. As surprising as it may seem, it explains the difficulty of prove this problem.

5. CONCLUSION

In this paper we establish Diophantine proofs of Abel's conjecture, Fermat Last Theorem for the exponents even, non-prime exponent, the first case and the second case $z \equiv 0\ (mod\ p)$. We analyse these proofs and establish a ranking in order of increasing difficulty in solving the Fermat problems treated.In perspective, we intend to:

- establish a Diophantine proof of the second remaining cases, i.e. to prove that if $xy \equiv 0\ (mod\ p)$ then $x^p + y^p \neq z^p$.
- extend methods to broader classes of equations: Catalan's equation, Beal problem and others General Fermat problem.
- introduce new concepts such as the universe and Diophantine galaxies, as well as the similarity principle, and then find applications for them in astronomy, astrophysics, cosmology and artificial intelligence.

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