

# NEW METHOD OF SIGNAL DENOISING BY THE PAIRED TRANSFORM

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## ABSTRACT

*A parallel restoration procedure obtained through a splitting of the signal into multiple signals by the paired transform is described. The set of frequency-points is divided by disjoint subsets, and on each of these subsets, the linear filtration is performed separately. The method of optimal Wiener filtration of the noisy signal is considered. In such splitting, the optimal filter is defined as a set of sub filters applied on the splitting-signals. Two new models of filtration are described. In the first model, the traditional filtration is reduced to the processing separately the splitting-signals by the shifted discrete Fourier transforms (DFTs). In the second model, the not shifted DFTs are used over the splitting-signals and sub filters are applied. Such simplified model for splitting the filtration allows for saving  $2N - 4(r + 1)$  operations of complex multiplication, for the signals of length  $N = 2^r$ ,  $r > 2$ .*

## KEYWORDS

*Time-frequency analysis, signal reconstruction, paired transform, Wiener filtration*

## 1. INTRODUCTION

Denoising of a time series signal on multiple channels is possible by splitting it into many signals which can be processed independently. First and foremost available tool is the Fourier integral transformation which represents a time signal in frequency bins/objects where time information is lost completely. Suppressing the noisy frequency objects is the common approach in this method. The wavelet transformation represents a time signal as scale-time signals where the information of scale (referred to as “frequency”) and time is preserved. These signals offer faster and better edge preservation algorithms than the Fourier method. Denoising as classified among a class of restoration problems has the objective to reconstruct the signal as close as possible to the actual signal in some distance measure. Restoration of a signal from garbled/incomplete information is the classical Wiener problem [1]. It solves primarily for a simple specification of the linear dynamical system,  $H \in C^{m \times n}$ ,  $m < n$ , which generates the wide-sense stationary random process  $x \in C^m$ . The solution of  $H\hat{z} = x$  is obtained by the *minimum-mean squares error* method which is the optimization of the  $l_2$ -norm  $\|z - \hat{z}\|^2$ , where  $z \in C^m$  is the actual signal. For a class of wide sense stationary random processes, the linear time invariant system matrix is a circulant-Toeplitz, so the computation is fast with the fast Fourier transform. Other fast approaches, which are based on the wavelet coefficients thresholding, introduce considerable artefacts [2]-[5]. Dohono's reconstruction by soft-thresholding of wavelet coefficients is at least as smooth as the minimum mean squares solution, therefore it removes several of these artefacts. Current parallel methods for signal restoration or filtration rely on *splitting* of a complex problem into partial problems whose independent solutions lead to a list of feasible solutions, or one final solution. The restoration problem is re-formulated as a set theoretic constrained optimization [7],[8]. The solution space is usually a Hilbert space, where the original signal is described solely by a family of constraints arising from a priori knowledge about the problem and from the observed data. We

also mention the parallel approaches such as block-iterative surrogate constraint splitting methods for solving a quadratic minimization problem under convex constraints [9].

In this paper, we consider the transformation of the signal  $\{f_n\}$  of length  $N = 2^r$ ,  $r > 2$ , into a set of splitting-signals  $\{f'_{p,pt}; t = 0: N/(2p) - 1\}$ ,  $p = 2^k$ ,  $k = 0:r$ . This is a frequency ( $p$ ) and time ( $t$ ) representation of the signal, which is accomplished by the fast paired transform [11]-[17]. The linear filtration of the noisy signal, including the optimal (Wiener) filtration, is described. Such filtration as the linear convolution can be reduced to filtration of splitting-signals in two different ways. In the first way, the Fourier transform method of filtration is reduced to calculation of the transform of each splitting-signal and filtration on the shifted grid of frequency-points. In other words in the traditional Fourier method, the shifted DFTs over the splitting-signals are calculated. All values of the filter function are reordered into  $r$  subsets and used for filtering the splitting-signals. These sub-filter functions or other such functions can also be used for denoising the signal in the frequency domain with the points on the non shifted grid. In this second way of filtration, one can save a large number of operations of complex multiplication and design new linear filters. Such filters can be transformed into equivalent filters in the shifted grid, which lead to the same results of filtration.

## 2. PAIRED REPRESENTATION OF THE SIGNAL

Given signal  $f_n$  and noise  $n_n$ , we consider the linear signal-independent-noise model  $g_n = f_n + n_n$ ,  $n = 0:(N - 1)$ . In paired representation, the signals  $f_n$ ,  $n_n$  and  $g_n$  are described by the corresponding sets of short splitting-signals of lengths  $N/2, N/4, \dots, 2, 1$ , and 1. For instance, the signal  $f_n$  is transformed into the following set of splitting-signals:

$$\chi'_N \rightarrow \begin{cases} \{f'_{1,t}; t = 0:(N/2 - 1)\} \\ \{f'_{2,2t}; t = 0:(N/4 - 1)\} \\ \{f'_{4,4t}; t = 0:(N/8 - 1)\} \\ \dots \dots \dots \\ \{f'_{N/4,0}, f'_{N/4,N/4}\} \\ \{f'_{N/2,0}\} \\ \{f'_{0,0}\}. \end{cases}$$

The 1-D  $N$ -point discrete paired transform  $\chi'_N$  is defined by the complete system of paired functions [11][12]:

$$\chi'_{p,pt}(n) = \begin{cases} 1; & \text{if } n = t \bmod N/p, \\ -1; & \text{if } n = (t + N/(2p)) \bmod N/p, \quad t = 0:(N/(2p) - 1), \\ 0; & \text{otherwise,} \end{cases} \quad (1)$$

where  $p = 2^k$ ,  $k = 0:(r - 1)$  and  $\chi'_{0,0}(n) \equiv 1$ ,  $n = 0:(N - 1)$ . The double numbering of the paired functions  $\chi'_{p,pt}(n)$  refers to the frequency-point ( $p = 2^k$ ) and time ( $pt$ ) which runs the interval of time points  $\{0, 1, 2, \dots, N/2 - 1\}$  with the step ("velocity") equals  $p$ . The paired transform is the frequency-time representation of the signal.

The components of the splitting-signal  $\{f'_{p,0}, f'_{p,p}, f'_{p,2p}, f'_{p,3p}, \dots, f'_{p,N/2-p}\}$  of the signal  $f_n$  are calculated by

$$f'_{p,pt} = \chi'_{p,pt} \circ f_n = \sum_{n=0}^{N-1} \chi'_{p,pt}(n) f_n, \quad t = 0: (N/(2p) - 1).$$

The last one-point splitting-signal is  $\{f'_{0,0}\} = f'_{0,0} = f_0 + f_1 + \dots + f_{N-2} + f_{N-1}$ .

*Example 1 (Case N=8)* The matrix of the eight-point discrete paired transform (DPT) is defined as

$$[\chi'_8] = \begin{bmatrix} [\chi'_{1,0}] \\ [\chi'_{1,1}] \\ [\chi'_{1,2}] \\ [\chi'_{1,3}] \\ [\chi'_{2,0}] \\ [\chi'_{2,2}] \\ [\chi'_{4,0}] \\ [\chi'_{0,0}] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The normalized coefficients  $\{\sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}, 2, 2, 2\sqrt{2}, 2\sqrt{2}\}$  of rows can be considered, to make this transform unitary. The first four basis paired functions correspond to the frequency  $p = 1$ , the next two functions correspond to frequency  $p = 2$ , and the last two functions correspond to the frequencies  $p = 4$  and  $0$ , respectively. The process of composition of these functions from the corresponding cosine waves defined in the interval  $[0,7]$  is illustrated in Figure 1.

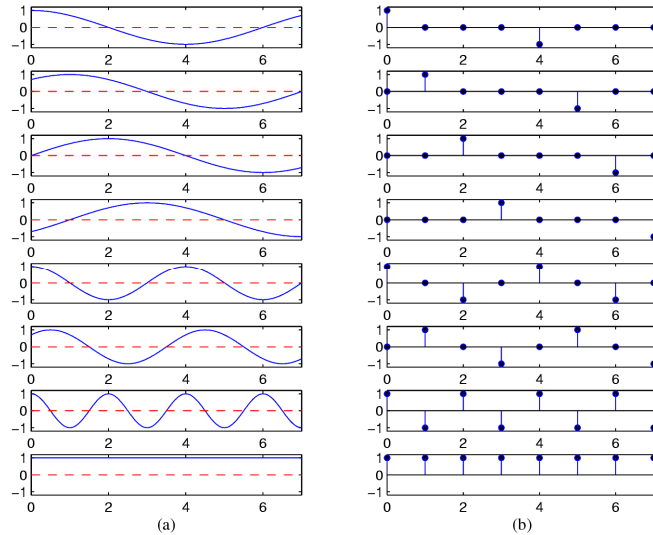


Fig. 1. (a) Cosine waves and (b) discrete paired functions of the eight-point DPT

Let  $f_n$  be the signal  $\{1,2,2,4,5,3,1,3\}$ . The paired transform of this signal results in the following four splitting-signals:

$$\chi'_N \rightarrow \begin{cases} \{f'_{1,0}, f'_{1,1}, f'_{1,2}, f'_{1,3}\}, \\ \{f'_{2,0}, f'_{2,2}\}, \\ \{f'_{4,0}\}, \\ \{f'_{0,0}\} \end{cases} = \begin{cases} \{-4, -1, 1, 1\}, \\ \{3, -2\}, \\ \{-3\}, \\ \{21\}. \end{cases}$$

Splitting-signals carry the spectral information of the signal  $f_n$  in different and disjoint subsets of frequency-points. In paired representation, the splitting of the  $N$ -point DFT

$$F_p = \sum_{n=0}^{N-1} f_n W_N^{np}, \quad p = 0:(N-1), \quad W_N = \exp(-j2\pi/N),$$

by  $(r+1)$  short DFTs is based on the following formula [16]:

$$F_{(2m+1)p} = \sum_{t=0}^{N/2p-1} (f'_{p,tp} W_{N/p}^t) W_{N/(2p)}^{mt}, \quad m = 0:(N/(2p)-1). \quad (2)$$

For the set of frequency-points  $\{p = 2^k; k = 0:(r-1)\}$ , the disjoint subsets

$$T'_p = \{(2m+1)p \bmod N; m = 0:(N/(2p)-1)\}$$

together with  $T'_0 = \{0\}$  divide the set of  $N$  frequency-points  $X_N = \{0, 1, 2, \dots, N-1\}$ . Therefore, the splitting-signal generated by the frequency  $p = 2^n$  is denoted by

$$f_{T'_p} = \{f'_{p,0}, f'_{p,p}, f'_{p,2p}, f'_{p,3p}, \dots, f'_{p,N/2-p}\}.$$

## 2.1. The first and second splitting-signals

To describe the meaning of equation (2), we first consider the  $p = 1$  case, when the first splitting-signal carries the spectral information of the signal at all odd frequency-points,

$$F_{2m+1} = \sum_{t=0}^{N/2-1} (f'_{1,t} W_N^t) W_{N/2}^{mt}, \quad m = 0:(N/2-1). \quad (3)$$

The  $N/2$ -point DFT over the modified splitting-signal

$$\tilde{f}_{T'_1} = \{f'_{1,0}, f'_{1,1} W^1, f'_{1,2} W^2, f'_{1,3} W^3, \dots, f'_{1,N/2-1} W^{N/2-1}\}$$

equals the  $N$ -point DFT of the original signal at odd frequency-points. It should be noted, that the right part of equation (3) is referred to as the  $N/2$ -point shifted DFT (SDFT) on the set of frequency-points  $\{m+1/2; m = 0:(N/2-1)\}$ , which we denote by

$$F_{m+1/2}^{(1)} = \sum_{t=0}^{N/2-1} f'_{1,t} W_{N/2}^{t(m+\frac{1}{2})}, \quad m + \frac{1}{2} = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots, \frac{N-1}{2}. \quad (4)$$

Thus, the  $N/2$ -point shifted DFT of the first splitting-signal coincides with the odd-indexed

DFT values of the original signal,  $F_{2m+1} = F_{m+1/2}^{(1)}$ ,  $m = 0:(N/2-1)$ .

For the  $p = 2$  case, the splitting-signal is  $f_{T'_2} = \{f'_{2,0}, f'_{2,2}, f'_{2,4}, f'_{2,6}, \dots, f'_{2,N/2-2}\}$  and

$$F_{(2m+1)2} = \sum_{t=0}^{N/4-1} (f'_{2,2t} W_{N/2}^t) W_{N/4}^{mt}, \quad m = 0:(N/4-1). \quad (5)$$

The  $N/4$ -point DFT over the modified splitting-signal  $\tilde{f}_{T'_2} = \{f'_{2,0}, f'_{2,2}W^2, f'_{2,4}W^4, f'_{2,6}W^6, \dots, f'_{2,N/2-2}W^{N/2-2}\}$  equals the  $N$ -point DFT of the original signal at all frequency-points which are odd-integer multiple of 2. The right part of equation (5) is referred to as the shifted  $N/4$ -point DFT on the set of frequency-points  $\{m + 1/2; m = 0: (N/4 - 1)\}$ , which is denoted by

$$F_{m+1/2}^{(2)} = \sum_{t=0}^{N/4-1} f'_{2,2t} W_{N/4}^{t(m+1/2)}, \quad m = 0: (N/4 - 1). \quad (6)$$

In the general case  $p = 2^k$  where  $k < r$ , the  $N$ -point DFT of the signal  $f_n$  at frequency-points of  $T'_p$  is the  $N/(2p)$ -point shifted DFT of the splitting-signal  $f_{T'_p}$ , i.e.,

$$F_{(2m+1)p} = F_{m+1/2}^{(p)}, \quad m = 0: N/(2p) - 1.$$

Thus, in paired representation, when the signal is described by the set of splitting-signals, the DFT of the signal  $f_n$  is described by a set of shifted DFTs of the splitting-signals. The shifts of frequency-points are by 0.5.

### 2.2. Discrete Linear Filters

In this section, we consider the traditional method of DFT for convoluting the signals. Let  $Y_p$ ,  $p = 0: (N - 1)$ , be the frequency characteristic of a given linear filter which describes the circular convolution of the noisy signal  $g_n = f_n + n_n$  with the impulse response of the filter,  $\hat{f}_n = y_n \otimes g_n$ . We denote by  $G_p$  and  $\hat{F}_p$  the  $N$ -point DFTs of the noisy and filtered signals, respectively. Then  $\hat{F}_p = Y_p G_p$ , for  $p = 0: (N - 1)$ . The block-diagram of the linear filtration of the noisy signal is given in Figure 2, where RDFT stands for the reversible DFT.

It follows directly from equation (2) that the process of filtration can be reduced to separate filtration of splitting-signals, as shown in Figure 3. The  $N/(2p)$ -point shifted DFTs of the splitting-signals  $f'_{p,pt}$  are multiplied by the corresponding filter functions  $Y_{(2m+1)p}$ , and then, the reversible SDFTs are calculated to obtain the filtered splitting-signals.

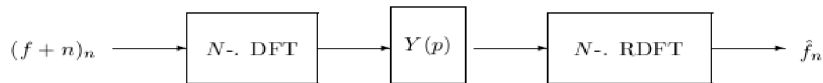


Fig. 2. Block-diagram of the Fourier transform-based method of filtration of the noisy signal.

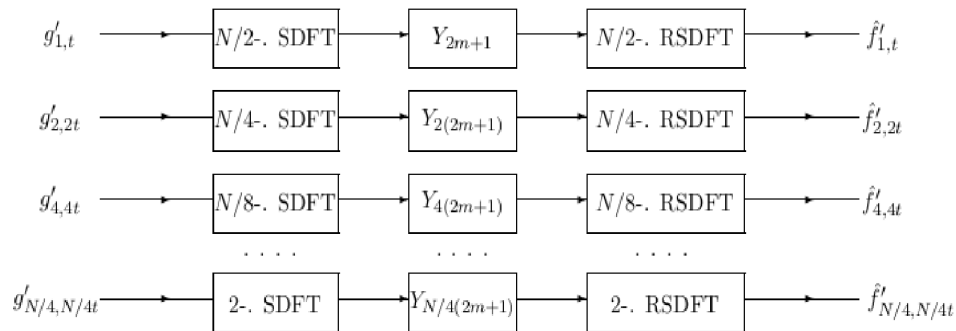


Fig. 3. Block-diagram of the Fourier transform-based method of filtration by splitting-signals.

The last two one-point splitting-signals  $g'_{N/2,0}$  and  $g'_{0,0}$  are not shown in the block diagram. The processing of these splitting-signals is reduced to one multiplication for each. The convoluted signal  $\hat{f}_n$  is calculated as the inverse  $N$ -point paired transform of the filtered splitting-signals. This convolution can be obtained by calculating the  $N$ -point reversible DFT over the transform  $G_p, p = 0:(N - 1)$ . Thus, the frequency characteristics are permuted onto  $(r + 1)$  subsets as  $\{Y_p; p = 0:(N - 1)\} \rightarrow \{Y_{(2m+1)p \bmod N}; m = 0:N/(2p) - 1; p = 1,2,4,8, \dots, N/2, Y_0\}$ . The filter is considered as the set of  $(r + 1)$  sub-filters.

*Example 2:* Consider the signal  $f_n$  of length  $N = 256$ , which is shown in Figure 4(a). The paired transform of the signal, as the set of splitting-signals  $\{f_{T'_1}, f_{T'_2}, f_{T'_4}, f_{T'_8}, \dots, f_{T'_{128}}, f_{T'_0}\}$  is shown in part b. The vertical dashed lines separate the first seven splitting-signals.

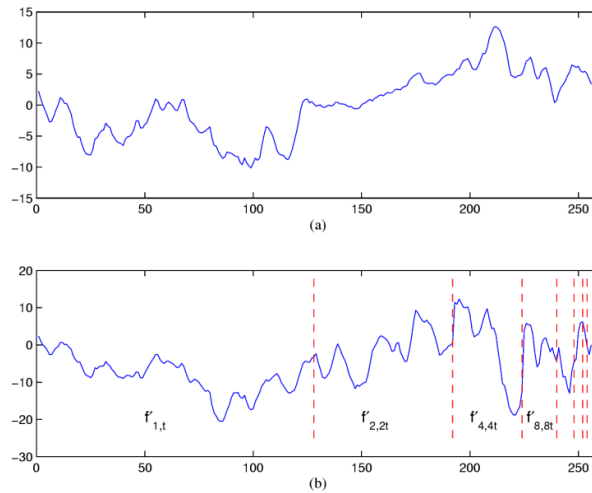


Fig. 4. (a) The signal of length 256 and (b) the 256-point discrete paired transform.

Figure 5 shows the 256-point DFT of the signal in absolute mode in part a. In part b, the same DFT is shown, but in the order that corresponds to the partition of the frequency-points  $\{0,1,2, \dots, 255\}$  by subsets  $T'_1, T'_2, T'_4, \dots, T'_{128}$ , and  $T'_0$ . The first 128 values corresponds to the 128-point DFT of the modified splitting-signal  $f'_{1,t} W^t$  which is also the 128-point SDFT of the splitting-signal  $f'_{1,t}$ . The next 64 values corresponds to the 64-point DFT of the modified splitting-signal  $f'_{2,2t} W^t$ , which is also the 64-point SDFT of the splitting-signal  $f'_{2,2t}$ , and so on.

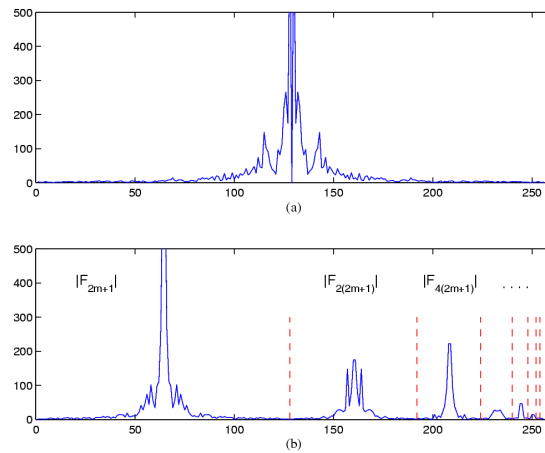


Fig. 5. (a) DFT of the signal and (b) DFTs of the modified splitting-signals. (The transforms are shown in absolute mode.)

We consider the smoothing of the signal by the circular convolution with the impulse response  $\{y_n\} = \{0,0,\dots,0,1,2,3,5,3,2,1,0,0,\dots,0\}/17$  with value  $y_0 = 5$ . Figure 6 shows the frequency characteristic  $Y_p$  of  $y_n$  in absolute mode in part a. In part b, this characteristic is shown in the same order as  $F_p$  in part b of Figure 5. These characteristics are used to filter the corresponding modified splitting-signals and they are separated in the graph by vertical lines.

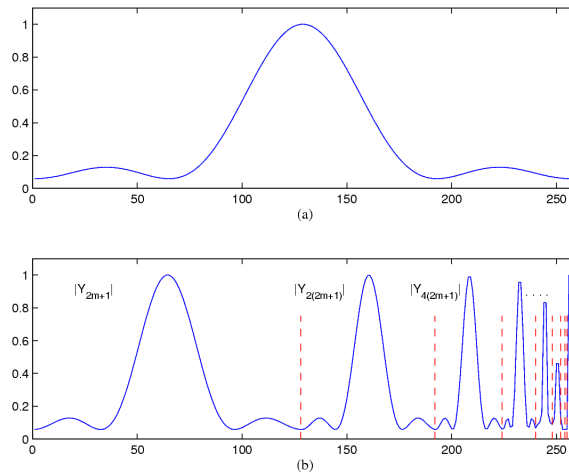


Fig. 6. (a) DFT of the filter and (b) the reordered filter (both in absolute scale).

The result of convolution of the signal,  $\hat{f}_n = f_n \otimes y_n$ , is shown in Figure 7 in part a, along with nine splitting-signals which were filtered separately in b. Together these splitting-signals define the filtered signal  $\hat{f}_n$  in a. In other words, the 256-point inverse paired transform over the set of these splitting-signals  $\{\hat{f}_{T'_1}, \hat{f}_{T'_2}, \hat{f}_{T'_4}, \hat{f}_{T'_8}, \dots, \hat{f}_{T'_{128}}, \hat{f}_{T'_0}\}$  is the signal  $\hat{f}_n$ . The following equation describes the inverse paired transform [14][15],[17]:

$$\hat{f}_n = \sum_{k=0}^{r-1} \frac{1}{2^{k+1}} \hat{f}'_{2^k, 2^k n \bmod N/2} + \frac{1}{N} \hat{f}'_{0,0}, \quad n = 0: (N - 1). \quad (7)$$

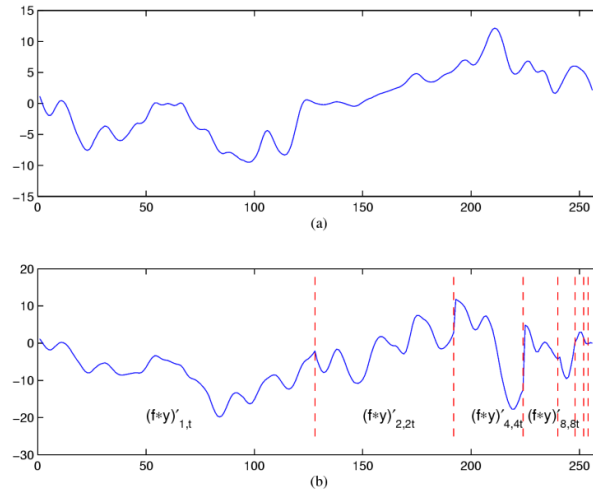


Fig. 7. (a) Filtered signal and (b) filtered splitting-signals.

All signals in part (a) and (b) are smooth. The last two one-point signal have not been changed, because the filter was defined with  $\sum y_n = 1$ . Thus the process of filtration of the original signal by the discrete Fourier transform was reduced to separate processing of eight splitting-signals. These signals were processed by the shifted DFTs.

### 3. NEW SIMPLIFIED SCHEME OF THE SIGNAL FILTRATION

It should be noted, that the calculation of the  $N$ -point DFT by the DFTs of the splitting-signals corresponds to the paired FFT with  $\mu(N) = N/2(r - 3) + 2$  operations of multiplication. The same number of multiplications is required for the reversible DFT. If we propose the process of filtration of each splitting-signal by the regular (not shifted) DFT, in other words, without modifying the splitting-signal by the twiddle-coefficients  $W^t$ , then many multiplications could be saved. Indeed, let us consider two types of filtration of the first splitting-signal with the block-diagrams shown in Figure 8.

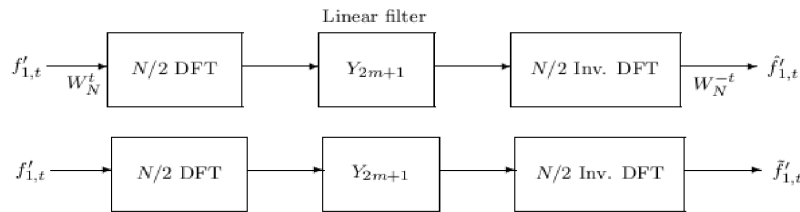


Fig. 8. Block-diagrams of filtration of the first splitting-signal.

In the first diagram, the splitting-signal is processed in the traditional way, with the  $N/2$ -point shifted DFT, which requires multiplications by the factors  $W_N^t, t = 0:(N/2 - 1)$ , and then, the calculation of the  $N/2$ -point DFT. In the second diagram, the same filter is used but the shifted DFTs are substituted by the DFTs, and therefore, it allows for saving  $N/2$  operations of multiplication by the twiddle factors. To be more accurate, this number is  $N/2 - 2$ , because of two trivial multiplications by  $W_N^0 = 1$  and  $W_N^{N/4} = -j$ . The same number of multiplications is saved when using the reversible  $N/2$ -point DFT instead of the shifted DFT.



We now consider the complete block-diagram of the simplified signal filtration without shifted DFTs, as shown in Figure 9.

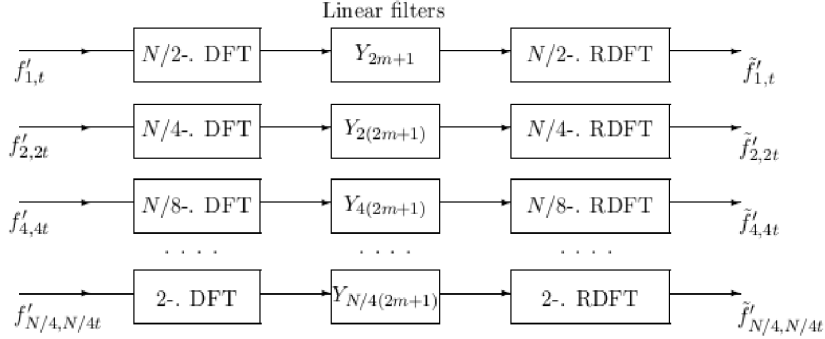


Fig. 9. Block-diagram of the filtration by splitting-signals.

The filtration of the signal by this method saves the following number of operations of complex multiplication:

$$\begin{aligned} \Delta\mu(N) &= 2[(N/2 - 2) + (N/4 - 2) + (N/8 - 2) + \dots + (8 - 2)] \\ &= 2[(N - 8) - 2(r - 3)] = 2N - 4r - 4. \end{aligned}$$

For instance when  $N = 256$ , such saving equals  $\Delta\mu(256) = 480$ . For the method of convolution with the block-diagram in Figure 3, the total number of required operations of multiplication equals  $2\mu(N) + N = N(r - 2) + 4$ . For the  $N = 256$  case, this number is 1540, and the saving in operations of multiplication is almost 32%.

### 3.1. Convolution of Splitting-Signals

To describe the process of filtration of the splitting-signals in the frequency-time domain, we consider in detail this operation over the first-splitting signal. On the set of frequency-points  $T_1$ , the application of the filter to the noisy signal  $g_n$  is described by

$$\hat{F}_{2m+1} = G_{m+1/2}^{(1)} Y_{m+1/2}^{(1)}, \quad m = 0 : (N/2 - 1), \quad (8)$$

where we denote the shifted DFTs by  $G_{m+1/2}^{(1)} = G_{2m+1}$  and  $Y_{m+1/2}^{(1)} = Y_{2m+1}$ .

Let  $\hat{f}_n$  be the filtered signal and  $\hat{f}'_{1,t}$  be its first splitting-signal, and let  $y'_{1,t}$  be the first splitting-signal of the impulse response of the filter  $Y_p$ . Then, the following calculations hold:

$$\begin{aligned} \hat{f}'_{1,t} W^t &= \frac{2}{N} \sum_{t=0}^{N/2-1} G_{m+1/2}^{(1)} Y_{m+1/2}^{(1)} W_{N/2}^{-tm} \\ &= \frac{2}{N} \sum_{m=0}^{N/2-1} \left[ \sum_{t_1=0}^{N/2-1} \left[ \sum_{s=0}^{N/2-1} g'_{1,s} W^s y'_{1,u} W^u \right] W_{N/2}^{t_1 m} \right] W_{N/2}^{-tm} \\ &\quad (u = (t_1 - s) \bmod N/2) \\ &= \frac{2}{N} \sum_{t_1=0}^{N/2-1} \left[ \sum_{s=0}^{N/2-1} g'_{1,s} W^s y'_{1,u} W^u \right] \sum_{m=0}^{N/2-1} W_{N/2}^{(t_1-t)m} \end{aligned}$$

$$\begin{aligned}
 &= \frac{2}{N} \sum_{t_1=0}^{N/2-1} \left[ \sum_{s=0}^{N/2-1} g'_{1,s} W^s y'_{1,u} W^u \right] \frac{N}{2} \delta(t_1 - t) \\
 &= \sum_{s=0}^{N/2-1} g'_{1,s} W^s y'_{1,(t-s) \bmod N/2} W^{(t-s) \bmod N/2}.
 \end{aligned}$$

Thus, we obtain the following:

$$\hat{f}'_{1,t} = \sum_{s=0}^{N/2-1} g'_{1,s} y'_{1,(t-s) \bmod N/2} W_N^{(s-t)} W_N^{(t-s) \bmod N/2}.$$

Since  $0 \leq t, s < N/2$  and  $(t-s) \bmod N/2 = (t-s)$ , if  $t \geq s$ , and  $(t-s) \bmod N/2 = (t-s) + N/2$ , if  $t < s$ , we have the following:

$$W_N^{(s-t)} W_N^{(t-s) \bmod N/2} = \text{sign}(t-s) = \begin{cases} 1, & \text{if } t-s \geq 0, \\ -1, & \text{if } t-s < 0. \end{cases}$$

Therefore,

$$\hat{f}'_{1,t} = \sum_{s=0}^{N/2-1} g'_{1,s} \text{sign}(t-s) y'_{1,(t-s) \bmod N/2}, \quad t = 0: (N/2 - 1). \quad (9)$$

We remind here, that the function  $y'_{1,t}$  as the function of  $t$  has the period  $N$  not  $N/2$ , and the function is ‘‘odd,’’ i.e.,  $y'_{1,-t} = -y'_{1,N/2-t}$ ,  $t = 1, 2, \dots, (N/2 - 1)$ . Equation (9) can be written as the linear (not circular) convolution calculated in  $N/2$  points:

$$\hat{f}'_{1,t} = \sum_{s=0}^{N/2-1} g'_{1,s} y'_{1,(t-s)}, \quad t = 0: (N/2 - 1). \quad (10)$$

The linear convolution is calculated only in the original time interval  $\{0, 1, \dots, N/2 - 1\}$  and can be calculated by the  $N$ -point circular convolution. We call this operation *the incomplete circular linear convolution of length  $N/2$* . Let  $\tilde{y}'_{1,t}$  be the extended function

$$\{\tilde{y}'_{1,t}; t = 0: (N - 1)\} = \{y'_{1,t}, t; t = 0: (N/2 - 1)\}, \{-y'_{1,N/2-t}; t = N/2: (N - 1)\}.$$

Let extended splitting-signal  $\tilde{g}'_{1,t}$  of length  $N$  be defined as

$$\{\tilde{g}'_{1,t}, t; t = 0: (N - 1)\} = \{g'_{1,t}; t = 0: (N/2 - 1)\}, \{0, 0, 0, \dots, 0\}.$$

It is not difficult to see, that the following holds:

$$\hat{f}'_{1,t} = \sum_{s=0}^{N/2-1} \tilde{g}'_{1,t} \tilde{y}'_{1,(t-s) \bmod N}, \quad t = 0: (N/2 - 1). \quad (11)$$

The linear convolution of the second and remaining splitting-signals with the filter is described similarly. Thus, the cyclic convolution of length  $N$  can be reduced to  $(r + 1)$  incomplete circular convolutions, when representing the signal in form of splitting-signals. As an example, we consider the signal of length 512 shown in part a of Figure 10. The paired transform as the set of ten splitting-signals is shown in b.

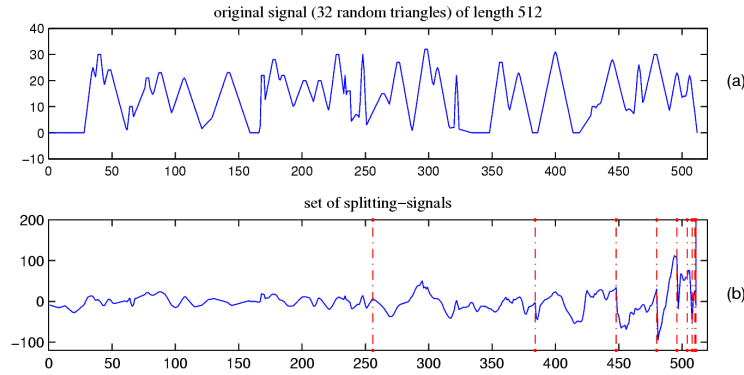


Fig. 10. (a) Random signal and (b) its splitting-signals.

Figure 11 in part a shows the signal  $g_n = f_n + n_n$  with an additive random Gaussian noise  $n_n$  which is normally distributed as  $N(0,17)$ . The first splitting-signal  $\{g'_{1,t}, t; t = 0:255\}$  of the noisy signal is shown in b, and the result of the optimal filtration of this splitting-signal in c. The result of the optimal filtration of the entire noisy signal  $g_n$  by ten splitting-signals is shown in d. The same result is achieved by using the Wiener filter over the noisy signal.

### 3.2. Parseval's Theorem and Paired Transform

The normalization of the basic paired functions as  $\chi'_{0,0} \rightarrow \chi'_{0,0}/\sqrt{N}$  and  $\chi'_{p,pt} \rightarrow \chi'_{p,pt}/\sqrt{2p}$ , when  $t = 0: (N/(2p) - 1), p = 1,2,4,8, \dots, N/2$ , leads to the unitary property of the paired transform. Therefore, according to Parseval's theorem, the difference of two signals  $f_n$  and  $\hat{f}_n$  in metric  $l_2$  can be written as

$$\begin{aligned} \varepsilon^2(f, \hat{f}) &= \sum_{n=0}^{N-1} (f_n - \hat{f}_n)^2 = \sum_{k=0}^{r-1} \frac{1}{2p} \sum_{t=0}^{N/(2p)-1} (f'_{p,pt} - \hat{f}'_{p,pt})^2 + \frac{1}{N} (f'_{0,0} - \hat{f}'_{0,0})^2 \\ &= \sum_{k=0}^{r-1} \frac{1}{2p} \varepsilon^2(f_{T'_p}, \hat{f}_{T'_p}) + \frac{1}{N} \varepsilon^2(f_{T'_0}, \hat{f}_{T'_0}) \end{aligned}$$

where we denote  $p = 2^k$ , and  $\varepsilon_p^2 = \varepsilon^2(f_{T'_p}, \hat{f}_{T'_p})$  are the differences of splitting-signals of  $f_n$  and  $\hat{f}_n$  in the same metric. The difference  $\varepsilon^2(f, \hat{f})$  can be referred to as the error of estimation, or filtration of the signal, and  $\varepsilon_p^2$  as the errors of filtration of the splitting-signals  $f_{T'_p}$ .

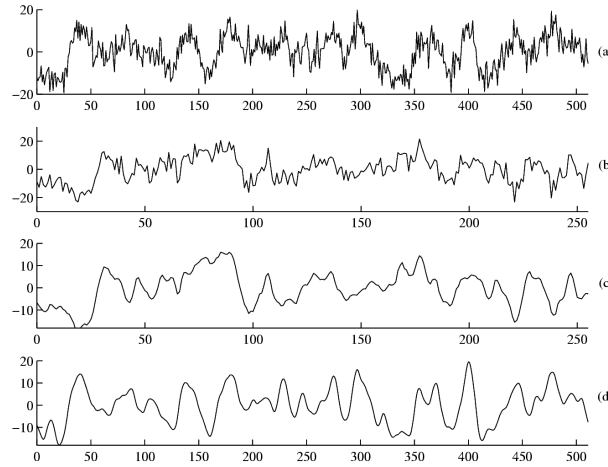


Fig. 11. (a) The noisy signal of length 512, (b) the noisy first splitting-signal of length 256, (c) filtered splitting-signal, and (d) the filtered signal.

It is clear that, the minimization of all errors  $\varepsilon_p^2$  may give not the best restoration  $\hat{f}$  of the signal. The splitting-signal can be processed separately, but they are not independent. All together they represent the signal  $f_n$ . They carry the information of the signal, which is transformed to them through the paired transform. On the other hand, the minimization of the error  $\varepsilon^2(f, \hat{f})$  provides good, if not the best minimum values for errors  $\varepsilon_p^2$  for the splitting-signals. The optimal filter as the set of coefficients,  $\{Y_p, p = 0:(N-1)\}$ , after being reordered as  $\{Y_{(2m+1)p}; m = 0:(N/(2p)-1), p = 2^k, k = 0:r\}$ , may be modified or used directly, to define the set of optimal, or close to optimal filters for splitting-signals. Our preliminary experimental results show that the set of these filters can be used for filtration of splitting-signals, if we want to save  $\Delta\mu(N) = 2N - 4r - 4$  operations of multiplication. We now consider the results of such filtration for the Wiener filter. There is a small difference in the results of the filtration of the signal  $g_n$ , by the Wiener filter over  $g_n$  and by the Wiener filters over splitting-signals.

### 3.3. Traditional Wiener filter (grid is shifted)

When denoising of the signal  $g_n = f_n + n_n$  is accomplished by the Wiener filter, the frequency characteristic of this filter is defined as

$$Y_p = \frac{\phi_f(p)}{\phi_f(p) + \phi_n(p)} = \frac{1}{1 + \phi_{n/f}(p)}, \quad p = 0:(N-1).$$

Here,  $\phi_f(p) = \langle |F_p|^2 \rangle$  and  $\phi_n(p) = \langle |N_p|^2 \rangle$  denote the power spectra of the original and noise signals, respectively. The notation  $\phi_{n/f}(p)$  is used for the noise-to-signal ratio,  $\phi_n(p)/\phi_f(p)$ , and  $\langle \cdot \rangle$  is used for the mathematical expectation of a random variable.

The Wiener filter results in a minimum error of signal reconstruction,  $\varepsilon^2(f, \hat{f}) = \min$ , where  $\hat{f}$  is the inverse Fourier transform of  $Y_p G_p$ . In paired representation, the Wiener filtration of the noisy signal  $g_n$  is reduced to separate processing of splitting-signals. Each splitting-signal  $g_{T'_p}$ ,  $p \in \{1, 2, 4, 8, \dots, N/2, 0\}$ , is modified by the cosine and sine waves and then the convolution with the "impulse response"  $y'_{p,pt}$  is calculated. Thus, in the frequency-time domain  $(p, pt)$ , the optimal filtration is achieved by amplifying the splitting-signal prior and after the convolution,

$$g_{T'_p} \rightarrow \mathbf{w} \cdot g_{T'_p} \rightarrow G_{(2m+1)p \bmod N} \rightarrow (YG)_{(2m+1)p \bmod N} \rightarrow \mathbf{w} \cdot \hat{f}_{T'_p} \rightarrow \hat{f}_{T'_p}.$$

Here the vector  $\mathbf{w}$  is  $(1, W_{N/p}^1, W_{N/p}^2, \dots, W_{N/p}^{N/2p})'$ . In terms of shifting DFT, this procedure is described by

$$g_{T'_p} \rightarrow G_{m+1/2}^{(p)} \rightarrow G_{m+1/2}^{(p)} Y_{m+1/2}^{(p)} \rightarrow \hat{f}_{T'_p}.$$

The filtration is accomplished by the  $N/(2p)$ -point discrete Fourier transform which is calculated in the shifted frequency grid  $\{m + 1/2; m = 0: (N/(2p) - 1)\}$ . The inverse paired transform,  $\{\hat{f}_{T'_p}\} \rightarrow \{\hat{f}_n\}$ , can be accomplished by the fast paired transform algorithm [15], or by the direct calculation (7).

#### 4. FILTRATION OF THE SPLITTING-SIGNALS ON THE NOT SHIFTED GRID

The filtration of splitting-signals described above is performed by the  $N/(2p)$ -point shifted DFTs, when the transforms are defined on the sets of frequency-points  $\{1/2, 1 + 1/2, 2 + 1/2, \dots, N/(2p) - 1/2\}$ . We now consider the DFTs of the splitting-signal in the grid  $\{0, 1, 2, \dots, N/2 - 1\}$ . In other words, for each frequency-point  $p = 2^k, k < r$ , together with this shifted DFT, we consider the  $N/(2p)$ -point DFT of the splitting-signal

$$F'_m = \sum_{t=0}^{N/(2p)-1} f'_{p,tp} W_{N/(2p)}^{mt}, \quad m = 0: (N/(2p) - 1). \quad (12)$$

Let  $g_n$  be the noisy signal,  $g_n = f_n + n_n$ . For the  $p = 1$  case, we consider the traditional grid  $\{m; m = 0: (N/2 - 1)\}$  of frequency-points and the  $N/2$ -point DFTs  $F'_m, N'_m$ , and  $G'_m = F'_m + N'_m$  of the first splitting-signals of  $f_n, n_n$ , and  $g_n$ , respectively. The block-diagram of optimal filtration of this splitting-signal by the Fourier transform which is calculated in the traditional frequency grid is given in Figure 12.

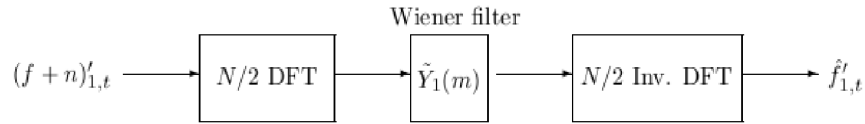


Fig. 12. Block-diagram of filtering the first splitting-signal in the grid  $\{m = 0: (N/2 - 1)\}$ .

If we assume a linear and time invariant model for the splitting-signal of the degraded signal  $g_n$ , then the frequency characteristic of the optimal filter in such model would be defined as follows:

$$\tilde{Y}_1(m) = \frac{\phi_{f'_{1,t}}(m)}{\phi_{f'_{1,t}}(m) + \phi_{n'_{1,t}}(m)}, \quad m = 0: (N/2 - 1).$$

Here  $\phi_{f'_{1,t}}(m)$  and  $\phi_{n'_{1,t}}(m)$  are the power spectra of the original and noise splitting-signals, respectively. Our preliminary results show, that this filter referred to as an optimal filter can be used effectively for processing the signal. The splitting-signal filtered by  $\tilde{Y}_1(m)$

$$\tilde{f}'_{1,t} = \frac{2}{N} \sum_{m=0}^{N/2-1} [\tilde{Y}_1(m) G'_m] W_{N/2}^{-mt}, \quad t = 0: (N/2 - 1),$$

differs from the splitting-signal obtained after the Wiener filtration

$$\hat{f}'_{1,t} = \frac{2}{N} \sum_{m=0}^{N/2-1} [Y_{m+1/2}^{(1)} G_{m+1/2}^{(1)}] W_{N/2}^{-(m+1/2)t}, \quad t = 0: (N/2 - 1),$$

on the shifted frequency grid  $\{m + 1/2; m = 0: (N/2 - 1)\}$ . The filter characteristic is calculated in the odd frequency-points by

$$Y_{m+1/2}^{(1)} = Y_{2m+1} = \frac{\phi_f(2m+1)}{\phi_f(2m+1) + \phi_n(2m+1)}, \quad m = 0: (N/2 - 1).$$

This characteristic differs from  $\tilde{Y}_1(m)$ , because the following holds:

$$\phi_{f'_{1,t}}(m) = \langle |F'_m|^2 \rangle \neq \langle |F_{m+1/2}^{(1)}|^2 \rangle = \langle |F_{2m+1}|^2 \rangle = \phi_f(2m+1),$$

and similarly  $\phi_{n'_{1,t}}(m) \neq \phi_n(2m+1)$ . Although the Wiener filter applied directly on the noisy splitting-signal has a larger error than does the Wiener filter on the original noisy signal, i.e.,  $\varepsilon^2(f_{T'_p}, \hat{f}_{T'_p}) \leq \varepsilon^2(f_{T'_p}, \tilde{f}_{T'_p})$ , the difference between these two errors is small. The similar reasoning can be used when processing the remaining splitting-signals by the corresponding characteristics  $\tilde{Y}_p(m)$ ,  $p = 2, 4, 8, \dots, N/2, 0$ . By splitting the noisy signal, the noise distribution does not change. Because the basic paired functions are populated by ones, the noise in the splitting-signals  $g'_{p,tp} = f'_{p,tp} + n'_{p,tp}$  is normally distributed. Figure 13 shows the optimal filtration of the first splitting-signal, when  $p = 1$ .

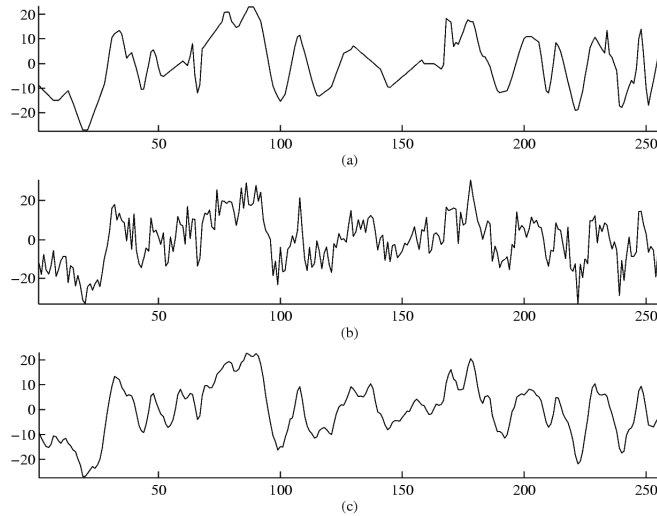


Fig. 13. (a) The first splitting-signal, (b) noisy splitting-signal, and (c) the reconstruction.

The result of restoration of the noisy signal  $f_n$  after processing all splitting-signals is shown in Figure 14, which is close to the direct Wiener filtration. The error of such filtration equals 0.103 (and 0.091 when using the direct Wiener filtration).

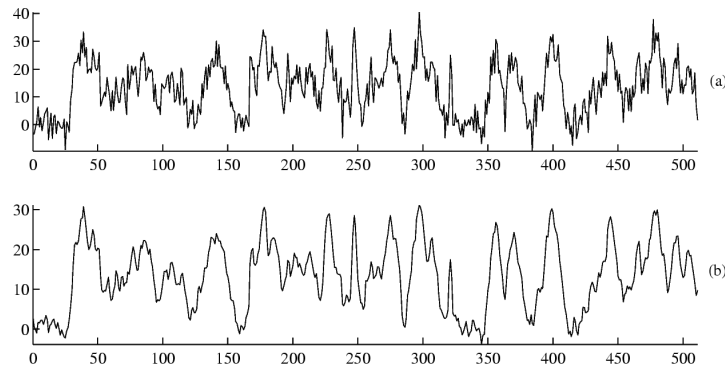


Fig. 14. (a) The random signal with Gaussian noise, and (b) the signal reconstruction.

Figure 15 shows another signal of length 256 in part a, along with the degraded signal in b, and signal restored by processing separately the splitting-signals in c. The error of such filtration equals 0.234 (and 0.239 when using the direct Wiener filtration).

In filtration, the Wiener filter is optimal which means the error  $\varepsilon^2(f, \hat{f})$  is minimum. The set of optimal filters for splitting-signals may not improve the result in the optimal filter over the original signal. However such filtration will be effective, because it allows for saving many operations of multiplication,  $2N - 4r - 4$ .

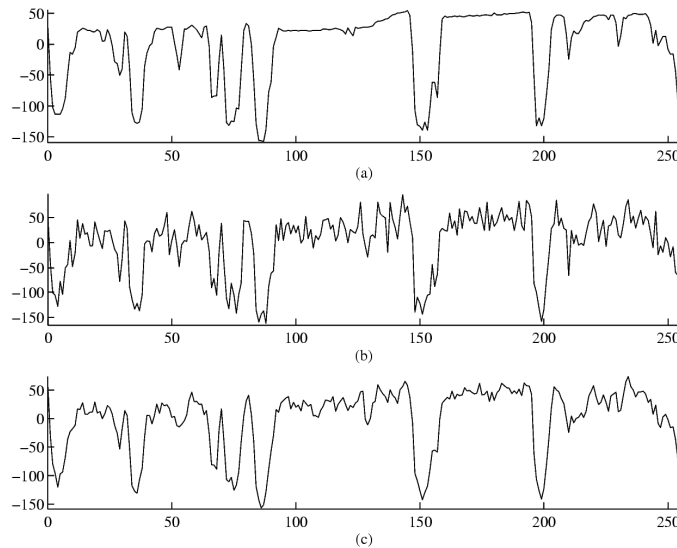


Fig. 15. (a) The random signal, (b) signal with Gaussian noise, and (c) the signal reconstruction.

## 5. OPTIMAL FILTRATION BY SPLITTING-SIGNALS: GENERAL MODEL

In this section, analytical equations are described for equivalent filter functions over the splitting-signals, when processing them in two models of filtration. Filter functions are called similar, or equivalent if they lead to the same results of signal filtration  $\hat{f}_n$ . We first describe the filtration of the splitting-signal  $\{g'_{1,t}; t = 0: (N/2 - 1)\}$  when this signal is modified and then filtered by the optimal Wiener sub-filter  $\{Y_1, Y_3, Y_5, \dots, Y_{N-1}\}$ . We also consider a such sub-filter with

$\{\tilde{Y}_1, \tilde{Y}_3, \tilde{Y}_5, \dots, \tilde{Y}_{N/2-1}\}$ , which leads to the same result over the splitting-signal, as the optimal Wiener sub-filter over the modified splitting-signal. The required for that equation

$$\left[ \sum_{t=0}^{N/2-1} g'_{1,t} W_{N/p}^{mt} \right] \tilde{Y}_m = \left[ \sum_{t=0}^{N/2-1} (g'_{1,t} W^t) W_{N/2}^{mt} \right] Y_{2m+1}$$

has the simple solution

$$\tilde{Y}_m = \left[ G_{m+1/2}^{(1)} / G_m^{(1)} \right] Y_{2m+1}, \quad m = 0: (N/2 - 1). \quad (13)$$

Here, we denote by  $G_{m+1/2}^{(1)}$  and  $G_m^{(1)}$  the shifted DFT and DFT of the noisy splitting-signal  $g'_{1,t}$ , respectively.

As an example, we consider the following linear model of the degraded signal:

$$g_n = f_n * h_n + n_n, \quad n = 0: (N - 1). \quad (14)$$

For simplicity of calculations, we consider that the random noise and original signal are not correlated. The Wiener filter is described by the function

$$Y_p = \frac{\bar{H}_p}{|H_p|^2 + \phi_{n/f}(p)}, \quad p = 0: (N - 1),$$

where  $H_p$  is the DFT of  $h_n$ , and  $\bar{H}_p$  denotes the complex conjugate of  $H_p$ . Figure 16 shows the original signal in part a. The smooth signal in b is the convolution of the signal with the impulse response  $h_n = [1, 2, 3, 2, 1]/9$ . The smooth signal with an additive noise is shown in c. The result of the Wiener filtration is given in d.

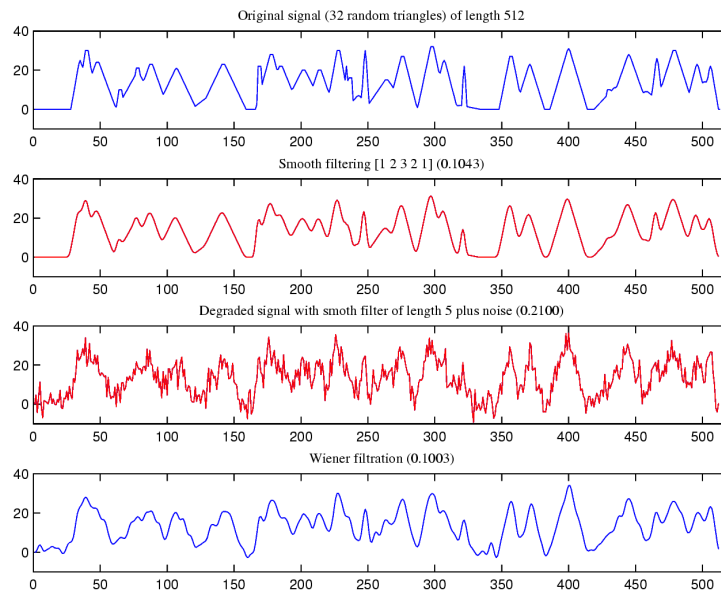


Fig. 16. The signals  $f_n$ ,  $f_n * h_n$ ,  $g_n$ , and filtered signal  $\hat{f}_n$  (from top to bottom).



Figure 17 shows the characteristics of the filter, namely, the function  $H_p$  in the decibel scale, the phase of  $H_p$ , the magnitude of the Wiener filter function, and the signal-to-noise ratio.

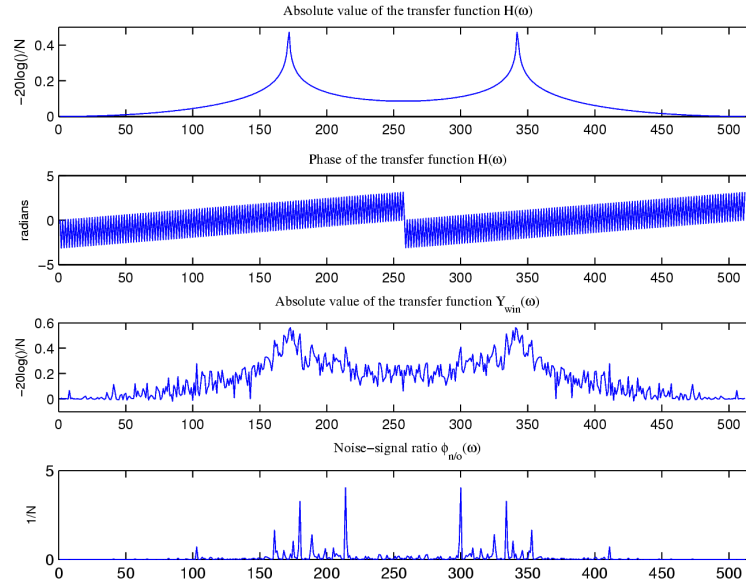


Fig. 17. Magnitude and phase of  $H_p$ ,  $|Y_p|$ , and signal-to-noise ratio (from top to button).

In paired representation, model (14) of the degraded signal is defined by the set of sub-models for splitting-signals. For the above example, Figure 18 shows the magnitudes and phases of these sub-filters. The signal-to-noise ratios for these sub-filters are also shown. All sub-filters are separated by the vertical lines in this drawing. As shown in Section III, each of these sub-models can be described by the incomplete circular convolution of the extended impulse response  $\tilde{h}_{p,pt}$  with the extended splitting-signal  $\tilde{g}_{p,pt}$  of the original signal plus the splitting-signal of the noise,

$$\hat{f}'_{p,pt} = \sum_{s=0}^{N/p-1} \tilde{g}_{p,ps} \tilde{h}_{p,p(t-s)\bmod N/p} + \tilde{n}_{p,pt}, \quad t = 0: (N/(2p) - 1).$$

For example, for the first sub-model when  $p = 1$ , we have the equation similar to (11),

$$\hat{f}'_{1,t} = \sum_{s=0}^{N-1} \tilde{g}_{1,s} \tilde{h}_{1,(t-s)\bmod N} + \tilde{n}_{1,t}, \quad t = 0: (N/2 - 1).$$

Given filter function  $\tilde{Y}_m$  for processing the first splitting-signal, the values of the equivalent filter  $Y_p$  at odd frequency-points are calculated as  $Y_{2m+1} = \left[ G_m^{(1)} / G_{m+1/2}^{(1)} \right] \tilde{Y}_m$ ,  $m = 0: (N/2 - 1)$ . Similar calculations are valid when considering the remaining splitting-signals. Figure 19 shows the magnitude of the filter functions  $\tilde{Y}_m$  and  $Y_{2m+1}$  in part a and b, respectively.

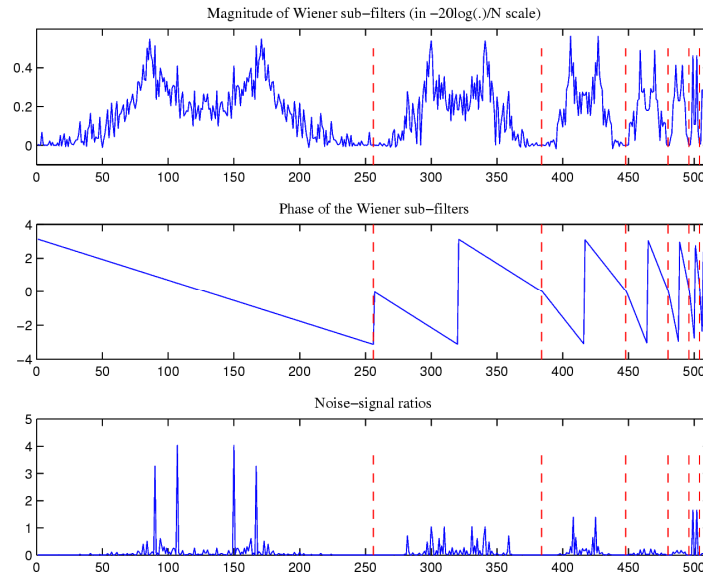


Fig. 18. Magnitudes and phases of sub-filters, and signal-to-noise ratios (from top to bottom).

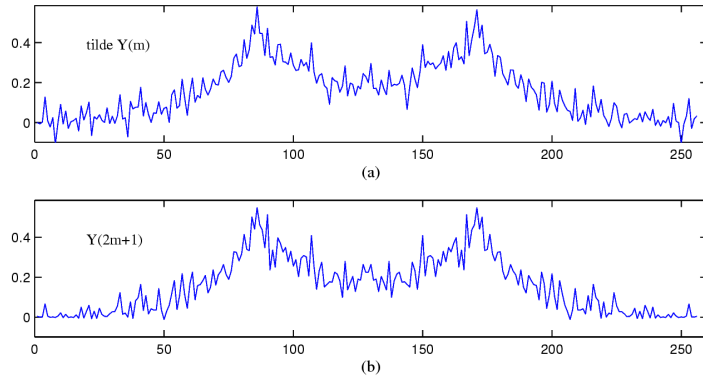


Fig. 19. The magnitudes of functions (a)  $\tilde{Y}_m$  and (b)  $Y_{2m+1}$ .

Let  $Y_p, p = 0: (N - 1)$ , be a filter function for processing the signal  $g_n$ . The filtration of the signal  $g_n$  is reduced to filtration of the modified splitting-signals  $\{g'_{p,pt} W_{N/p}^t; t = 0: (N/(2p) - 1)\}$ , by the sub-filters  $\{Y_{(2m+1)p}; m = 0: (N/(2p) - 1)\}$ , respectively. Each of these operations describes the filtration of the corresponding splitting-signal in the shifted frequency grid, because  $Y_{(2m+1)p} = Y_{m+1/2}^{(p)}$ . The equivalent sub-filters  $\tilde{Y}_m = \tilde{Y}_m^{(p)}$  for processing splitting-signals  $\{g'_{p,pt}; t = 0: (N/(2p) - 1)\}$  in the not shifted grid are calculated by  $\tilde{Y}_m^{(p)} = \left[ G_{m+1/2}^{(p)} / G_m^{(p)} \right] Y_{(2m+1)p}$ ,  $m = 0: (N/(2p) - 1)$ . For the second splitting-signal, Figure 20 shows the magnitudes of filter functions  $\tilde{Y}_m^{(2)}$  and  $Y_{2(2m+1)}$  in part a and b, respectively.

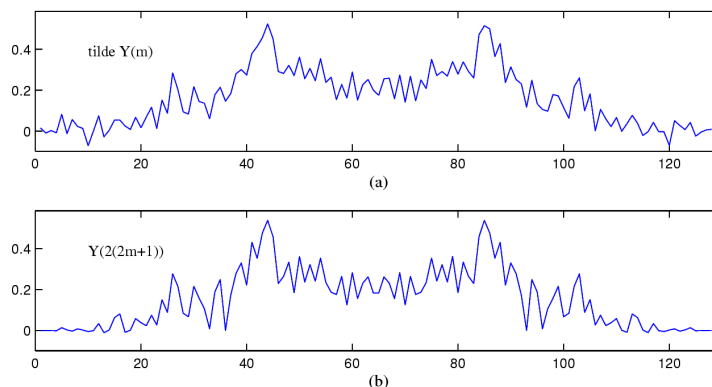


Fig. 20. The magnitudes of functions (a)  $\tilde{Y}_m^{(2)}$  and (b)  $Y_{2(2m+1)}$ .

Thus, we have described two different models of signal filtration through the splitting-signals when the filter is presented by the set of sub-filters. One model describes the separation of filtration by the modified splitting-signals, and another model describes the filtration by the splitting-signals without complex operation of modification, which requires multiplications by the twiddle coefficients. Equation 13 describes a way of finding the equivalent filters in both models.

## 6. CONCLUSIONS

The one-dimensional (1-D) signal can be filtered, by transforming the signal of length  $N = 2^r$ ,  $r > 2$ , into the set of splitting-signals of lengths  $N/2, N/4, \dots, 2, 1$ , and 1. These splitting-signals carry the spectral information of the original signal in disjoint subsets of frequency-points on both shifted and not shifted grids. The traditional filtration, including the optimal Wiener filtration, can be reduced to filtration of the splitting-signals on the shifted grid. In the time domain, this process is described by the incomplete circular convolutions of the splitting-signals with the corresponding characteristics of the filter. The filtration of the splitting-signals can also be performed on the not shifted grid, and such filtration allows to saving  $2(N - r - 2)$  operations of complex multiplication. The described models of signals filtration by splitting-signals can be generalized and used for the two-dimension (2-D) case of signals, or images. For the case when size of the image is  $N \times N$ , and  $2^r$ ,  $r > 2$ , the 2-D paired transform [11],[15] represents the image as the set of  $(3N - 2)$  splitting-signals of length  $N/2, N/4, \dots, 2$ , and 1. The filtration of the image can therefore be reduced to the filtration of the splitting-signals as described above for the 1-D case.

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