EDGE-NEIGHBOR RUPTURE DEGREE ON GRAPH OPERATIONS

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ABSTRACT

Vulnerability and reliability parameters measure the resistance of the network to disruption of operation after the failure of certain stations or communication links in a communication network. An edge subversion strategy of a graph G, say S, is a set of edge(s) in G whose adjacent vertices which is incident with the removal edge(s) are removed from G. The survival subgraph is denoted by $G - S$. The edge-neighbor-rupture degree of connected graph G, $ENR(G)$, is defined to be $ENR(G) = max\{\omega(G - S) - |S| - m(G - S): S \subseteq E(G), \omega(G - S) \geq 1\}$ where S is any edge-cut-strategy of G, $\omega(G - S)$ is the number of the components of $G - S$, and $m(G - S)$ is the maximum order of the components of $G - S$. In this paper we give some results for the edge-neighbor-rupture degree of the graph operations and Thorny graph types are examined.

KEYWORDS

Edge-neighbor-rupture degree, Thorny graphs, Vulnerability, Reliability.

1. INTRODUCTION

A communication network can be brokedown to pieces partially or completely from unexpected factors. This situation can prevent data transmit so there would be a big problem on the system to perform it's task. Therefore, the vulnerability and the reliability measure the resistance of the network disturbance of operations after the failure of certain stations. To measure the vulnerability and the reliability we have some parameters which are connectivity [7,11,12], integrity [3], scattering number [8], rupture degree [9], neighbor-rupture degree [1] and edge-neighbor-rupture degree [2].

Terminology and notations are not defined in this paper but it can be found [4,5]. Let $G = (V, E)$be a simple graph and let e be any edge of G. The set, $N(e) = \{f \in E(G) | e \neq f; e\ and\ f\ are\ adjacent\}$ is the open neighborhood of e, and $[e] = \{e\} \cup$ (e) is the closed neighborhood of e. An edge e in $G$is said to be subverted if the closed neighborhood of e is removed from G. In other words, if $e = \{u, v\}$than$G - [e] = G - \{u, v\}$. A set of edges $S = \{e_1, e_2, \ldots, e_m\}$is called an edge subversion strategy of Gif each of the edges in Shas been subverted from G. If Shas been subverted from the graph G, then the remaining graph is called survival graph, denoted by $G - S$. An edge subversion strategy $S$is called an edge-cut-strategy of Gif the survival subgraph $G - S$is disconnected or is a single vertex or the empty graph

[10].$S$is any edge-cut-strategy of G, $(G-S)$ is the number of the components of $G-S$, and $(G-S)$ is the maximum order of the components of $G-S$.

The definition of edge-neighbor-rupture degree of a connected graph G is

$ENR(G)=max\{\omega(G-S)-|S|-m(G-S): S\subseteq E(G), \omega(G-S)\geq 1\}$.

Definition 1.1:$r\in N^{+}$ and $\forall v\in V(G)$ for each vertex of G, if $\deg(v)=r$ than G is called *r-regular* graph [4].

Definition 1.2: G_1 and G_2 graphs have disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. The *join operation* of two G_1 and G_2 graphs is denoted by G_1+G_2 and consist of $G_1\cup G_2$which is union of two G_1 and G_2 graphs and all edges joining V_1 and V_2[4].

Definition 1.3: Let $p_1, p_2, \dots, p_n$be non-negative integers. The *Thorny Graph* of the graph G, with parameters$p_1, p_2, \dots, p_n$,is obtained by attaching p_i new vertices of degree one to the vertexu_i of the graph$G$, $i=1,2,\dots,n$.

The Thorny graph of the graph $G$is denoted by $G^{\star}$, or if the respectiveparameters need to be specified, by $G^{\star}(p_1, p_2, \dots, p_n)$[6].

Definition 1.4: Forthree or more disjoint graphs $G_1, G_2, \dots, G_n$, the *sequential join*

$G_1+G_2+\cdots+G_n$

is the graph

$(G_1+G_2)\cup(G_2+G_3)\cup\dots\cup(G_{n-1}+G_n)$[4].

Definition 1.5: Connectivity $k(G)$ is the minimum number of vertices that need to be removed in order to disconnect a graph [4].

Definition 1.6: The integrity of a graph $G=(V,E)$ is defined by $I(G)=min\{|S|+m(G-S)\}$; $S\subset V(G)$ where $m(G-S)$ denotes the order of largest component in $G-S$[3].

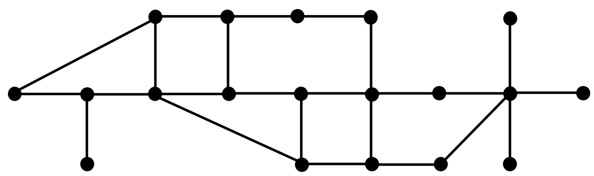
Definition 1.7: The rupture degree of a non-complete connected graph G is defined by $r(G)=max\{\omega(G-S)-|S|-m(G-S): S\subset V(G), \omega(G-S)>1\}$ where $\omega(G-S)$ denotes the number of components in the graph $G-S$ and $m(G-S)$ is the order of the largest component of $G-S$[9].

Definition 1.8: The neighbor integrity of a graph G is defined by

$NI(G)=min\{|S|+c(G-S): S\subset V(G)\}$where$S$ is any vertex subversion strategy of G and $c(G-S)$ is the order of the largest component of $G-S$[10].

Definition 1.9: The neighbor rupture degree of a non-complete connected graph G is defined to be $Nr(G)=max\{\omega(G-S)-|S|-c(G-S): S\subset V(G), \omega(G-S)\geq 1\}$ where S is any vertex subversion strategy of G, $\omega(G-S)$ is the number of connected components in $G-S$ and $c(G-S)$ is the maximum order of the components of $G-S$[13].

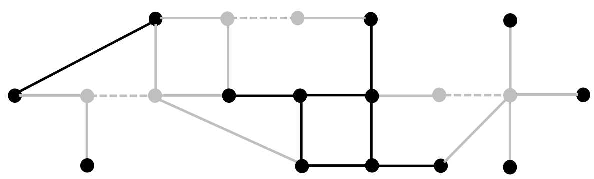
Definition 1.10: The edge-neighbor-rupture degree of a connected graph G is defined to be $ENR(G)=max\{\omega(G-S)-|S|-m(G-S) : S\subseteq E(G), \omega(G-S)\geq 1\}$, where S is any edge-cut-strategy of G, $\omega(G - S)$ is the number of the components of $G - S$, and $m(G - S)$ is the maximum order of the components of $G - S$.



Example: Let's give an exampleof the calculation of the edge-neighbor-rupture-degree.

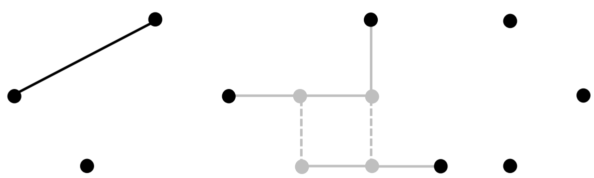
$\omega(G - S) = 1$
$|S| = 0$
$m(G - S) = 19$
$\omega(G - S) - |S| - m(G - S) = 1 - 0 - 19 = -18$

Figure 1.1



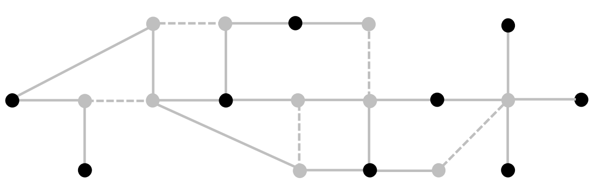
$\omega(G - S) = 6$
$|S| = 3$
$m(G - S) = 7$
$\omega(G - S) - |S| - m(G - S) = 6 - 3 - 7 = -4$

Figure 1.2



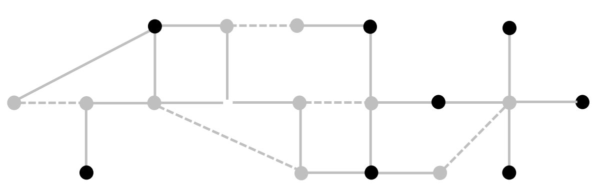
$\omega(G - S) = 8$
$|S| = 5$
$m(G - S) = 2$
$\omega(G - S) - |S| - m(G - S) = 8 - 5 - 2 = 1$

Figure 1.3



$\omega(G - S) = 9$
$|S| = 5$
$m(G - S) = 1$
$\omega(G - S) - |S| - m(G - S) = 9 - 5 - 1 = 3$

Figure 1.4



$\omega(G - S) = 9$
$|S| = 5$
$m(G - S) = 1$
$\omega(G - S) - |S| - m(G - S) = 9 - 5 - 1 = 3$

Figure 1.5

As shown, figure 1.1, figure 1.2,figure 1.3,figure 1.4 and figure 1.5

$ENR(G) = max\{-18, -4,1,3,3\}$

$ENR(G) = 3.$

In this paper, the edge-neighbor-rupture degree of some graphs is obtained and the relations between edge-neighbor-rupture degree and other parameters are determined [2].

2. EDGE-NEIGHBOR-RUPTURE DEGREE ON GRAPH OPERATIONS

In this section some theorems are given for edge-neighbor-rupture degree on the graph operations. Connected, undirected, simple graphs are examined.

Theorem 2.1: Let$G$ be a regular graph and $G^\star$ is a thorny graph of G (adding a vertex to any vertex of a graph). Then the edge-neighbor-rupture degree of G is,

$ENR(G^\star) = ENR(G) + 1.$

Proof: Since$G$ is regular graph, you can start from any edge to delete. There are two cases.

Case 1: If we start to delete an edge that is incident to an added vertex, while Sedge-cut-strategynumber is not changing,$(G–S)$ numbers of components are increased 1. So the result is increased 1.

Case 2: If we start to delete an edge that is not incident to an added vertex, while Sedge-cut-strategy number and the number of the components are not changed, maximum order of the components are increased. So the result is increased.

The result takes maximum value in case 1. So the proof is completed.■

Theorem 2.2: Edge-neighbor-rupture degree of thorny of regular graph G (adding i vertices equally to every vertex of a graph G for$1 \leq i \leq n$) is,

$ENR(G^\star) = ni - \left\lceil\frac{n}{2}\right\rceil - 1.$

Proof: Let S be anedge-cut-strategyof $G^\star$and let $|S| = r$. There are two cases for the elements of S.

Case 1: If $0 \leq r < \left\lceil\frac{n}{2}\right\rceil$, then $\omega(G^\star - S) \leq 2ri + 1$ and $m(G^\star - S) \geq i + 1$, so we have

$\omega(G^\star - S) - |S| - m(G^\star - S) \leq 2ri + 1 - r - (i + 1) = 2ri - r - i.$

Let's$(r) = 2ri - r - i$. Since $f(r)$is an increasing function in $0 \leq r < \left\lceil\frac{n}{2}\right\rceil$, it takes its maximum value at $\left(\left\lceil\frac{n}{2}\right\rceil - 1\right)$, and

$$f\left(\left\lceil\frac{n}{2}\right\rceil - 1\right) = 2\left(\left\lceil\frac{n}{2}\right\rceil - 1\right)i - \left\lceil\frac{n}{2}\right\rceil - i.$$

Thus we get $(G^\star) \leq 2\left(\left\lceil\frac{n}{2}\right\rceil - 1\right)i - \left\lceil\frac{n}{2}\right\rceil - i$.

Case 2: If $\left\lceil\frac{n}{2}\right\rceil \leq r \leq \frac{3n^2-n}{2}$, then $\omega(G^\star - S) \leq ni$and $m(G^\star - S) \geq 1$, so we have

$\omega(G^\star - S) - |S| - m(G^\star - S) \leq ni - r - 1.$

Let's$f(r) = in - r - 1$.Since $f(r)$is a decreasing function in $0 \leq r < \left\lceil\frac{n}{2}\right\rceil$, it takes its maximum value at $\left\lceil\frac{n}{2}\right\rceil$, and $f\left(\left\lceil\frac{n}{2}\right\rceil\right) = ni - \left\lceil\frac{n}{2}\right\rceil - 1.$

Thus we get $ENR(G^\star) \leq ni - \left\lceil\frac{n}{2}\right\rceil - 1.$

From Case1 and Case 2 we have

$$ENR(G^\star) \leq ni - \left\lceil\frac{n}{2}\right\rceil - 1. \quad (1)$$

There exist S^* such that

$r = \left\lceil\frac{n}{2}\right\rceil$, $\omega(G^\star - S^*) = ni$ and $m(G^\star - S^*) = 1$, thus we have

$$ENR(G^\star) \geq ni - \left\lceil\frac{n}{2}\right\rceil - 1. \quad (2)$$

From (1) and (2) we get

$$ENR(G^\star) = ni - \left\lceil\frac{n}{2}\right\rceil - 1.$$ ■

Corollary 2.1: Let$P_n^\star$ is a Thorny graph of P_n (adding i vertices equally to every vertex of a graph for$2 \leq i \leq n$). Then the edge-neighbor-rupture degree of $P_n^\star$ is,

$ENR(P_n^\star) = ni - \left\lceil\frac{n}{2}\right\rceil - 1.$

Proof: It is the same as proof of Theorem 2.2 ■

Corollary 2.2: Let$W_n^\star$ is a Thorny graph of W_n (adding i vertices equally to every vertex of a graph for$2 \leq i \leq n$). Then the edge-neighbor-rupture degree of $W_n^\star$ is,

$ENR(W_n^\star) = ni - \left\lceil\frac{n}{2}\right\rceil - 1.$

Proof: It is the same as proof of Theorem 2.2. ■

Theorem 2.3: Edge-neighbor-rupture degree of Thorny graphofS_n is$(i \geq 2)$,

$ENR(S_n^\star) = ni - n.$

Proof: Let S be an edge-cut-strategyof $S_n^\star$and let $|S| = r$. There are two cases for the elements of S.

Case 1: If $0 \leq r < n-1$, then $\omega(S_n^\star - S) \leq ri + 1$ and $m(S_n^\star - S) \geq i + 1$, so we have

$\omega(S_n^\star - S) - |S| - m(S_n^\star - S) \leq ri + 1 - r - (i+1) = ri - r - i.$

Let$f(r) = ri - r - i$. Since $f(r)$is an increasing function in $0 \leq r < n-1$, it takes its maximum value $(n-2)$ and $f(n-2) = (n-2)(i-1) - i$.

Thus we get$ENR(S_n^\star) \leq ni - n - 3i + 2$.

Case 2: If $n - 1 \leq r \leq n^2 + n - 1$ then $\omega(S_n^\star - S) \leq in$ and $m(S_n^\star - S) \geq 1$, so we have $\omega(S_n^\star - S) - |S| - m(S_n^\star - S) \leq ni - r - 1$.

Let $f(r) = ni - r - 1$.Since $f(r)$is a decreasing function in$n - 1 \leq r \leq n^2 + n - 1$it takes its maximum value at $(n-1)$ and

$f(n-1) = ni - n + 1 - 1 = ni - n.$

Thus we get $ENR(S_n^\star) \leq ni - n$.

From Case1 and Case 2 we have

$$ENR(S_n^\star) \leq ni - n. \tag{3}$$

There exist S^* such that

$r = n - 1$, $\omega(S_n^\star - S^*) = ni$ and $m(S_n^\star - S^*) = 1$, thus we have

$$ENR(S_n^\star) \geq ni - n. \tag{4}$$

From (3) and (4) we get $(S_n^\star) = ni - n$. ■

Theorem 2.4: Let P_n,P_m is a path of order n and m respectively. The edge-neighbor-rupture degree of addition of P_n and P_m is,

$$ENR(P_n + P_m) = \begin{cases} ENR(P_m) - \frac{n}{2}, & n \text{ is even} \\ ENR(P_{m-1}) - \left\lceil \frac{n}{2} \right\rceil, & n \text{ is odd} \end{cases}, \qquad n < m$$

Proof:

Case 1: If we select an edge-cut-strategy fromP_n, we needed $\frac{n}{2}$ edges to delete all the edges of P_n so$P_m$is remained. Therefore,

$$ENR(P_n + P_m) = \begin{cases} ENR(P_m) - \frac{n}{2}, & n \text{ is even} \\ ENR(P_{m-1}) - \left\lceil \frac{n}{2} \right\rceil, & n \text{ is odd} \end{cases}$$

Case 2: If we select an edge-cut-strategy from combining edges of P_n and P_m, We get,

$ENR(P_n + P_m) = ENR(P_{m-n}) - n.$

Case 3: If we select an edge-cut-strategy fromP_m, we need$\frac{n}{2}$ edges to delete all the edges of $P_m$soP_n is remained. Therefore,

$$ENR(P_n + P_m) = \begin{cases} ENR(P_n) - \frac{m}{2}, & m \text{ is even} \\ ENR(P_{n-1}) - \left\lceil \frac{m}{2} \right\rceil, & m \text{ is odd} \end{cases}$$

The results take maximum value in case 1. So the proof is completed.

Corollary 2.3: The edge-neighbor-rupture degree of P_2+P_n is,

$$ENR(P_2 + P_m) = ENR(P_n - 1).$$

Theorem 2.5: Let$G_1, G_2, \ldots, G_n$be connected graphs then,

$$ENR(G_1 \cup G_2 \cup \ldots \cup G_n) \geq ENR(G_1) + ENR(G_2) + \cdots + ENR(G_n)$$

Proof: Let$G = G_1 \cup G_2 \cup \ldots \cup G_n$be union of $G_1, G_2, \ldots, G_n$. Let $S = S_1, S_2, \ldots, S_n$ be an edge Nr-set of$G_1, G_2, \ldots, G_n$ respectively and let $S = S_1 \cup S_2 \cup \ldots \cup S_n$be an edge-subversion strategy of G. Then we obtain,

$$\begin{aligned} ENR(G) &\geq \omega(G - S) - |S| - m(G - S) \\ &= [\omega(G_1 - S_1) + \omega(G_2 - S_2) + \cdots + \omega(G_n - S_n)] - [|S_1| + |S_2| + \cdots + |S_n|] \\ &\quad - [m(G_1 - S_1) + m(G_2 - S_2) + \cdots + m(G_n - S_n)] \\ &= \omega(G_1 - S_1) - |S_1| - m(G_1 - S_1) + \omega(G_2 - S_2) - |S_2| - m(G_2 - S_2) + \cdots + \omega(G_n - S_n) \\ &\quad - |S_n| - m(G_n - S_n) \end{aligned}$$

$$ENR(G_1) + ENR(G_2) + \cdots + ENR(G_n).$$ ■

Theorem 2.6: Let G be a connected graph, then the edge-neighbor-rupture degree of G is,

$$ENR(G) \geq -\left\lfloor \frac{n}{2} \right\rfloor.$$

Proof: For the minimum value of $ENR(G)$, $\omega(G - S)$ must be the smallest, $|S|$ and $m(G - S)$ must be the greatest.

Let S be an edge-cut-strategy of G and $|S| = r$. There are two cases for the elements of S.

Case 1: If $r \leq \left\lfloor \frac{n-1}{2} \right\rfloor$then $\omega(G - S) = 1$ and$m(G - S) \geq n - 2r$.

So we have
$\omega(G - S) - |S| - m(G - S) \leq 1 - r - (n - 2r) \quad = 1 + r - n = f(r)$.

This equality takes maximum value for $r = \left\lfloor \frac{n-1}{2} \right\rfloor$.

There are two conditions:

Case i: If *n* is odd;

$$f(r) = 1 + \frac{n-1}{2} - n = -\frac{n-1}{2} = -\left\lfloor\frac{n}{2}\right\rfloor.$$

Case ii: If n is even;

$$f(r) = 1 + \frac{n-2}{2} - n = -\frac{n}{2}$$

Case 2: If $\frac{n(n-1)}{2} > r > \left\lfloor\frac{n-1}{2}\right\rfloor$ then $\omega(G-S) \leq 1$and $m(G-S) \geq 1$.

$\omega(G-S) - |S| - m(G-S) \leq 1 - r - 1 = -r.$

This equality takes the maximum value for $r = \left\lfloor\frac{n}{2}\right\rfloor$.

From all cases, we obtain,$ENR(G) \geq -\left\lfloor\frac{n}{2}\right\rfloor$. ■

Corollary 2.4: Let G be a connected graph, then the edge-neighbor-rupture degree of G is,

$-\left\lfloor\frac{n}{2}\right\rfloor \leq ENR(G) \leq n-4.$

Proof: From Theorem 2.6 and [2], the proof is complete. ■

3. COMPUTING EDGE-NEIGHBOR-RUPTURE DEGREE OF A GRAPH

In this section, an algorithm is proposed in order to calculate the edge-neighbor-rupture degreefor any simple finite undirected graph without loops and multiple edges by using the findENR function.

Algorithm Edge Neighbor Rupture (ENR)
Output: ENR value for given any graph *G*

Begin
    ENR←-∞;
    for all edge subsets$E_S \subseteq E$**do**
            if findENR(G, E_S)> ENR**then**
            ENR=findENR(G, E_S);
            end
    end
end.

The function below,find ENR, returns the ENR value for the edge subset for the graph.

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function findENR(G, E_S);
Input: Graph$G(V, E)$, edgesubsetE_S
Output:ENR value for given an $E_S$edge subset of G.

Begin
    V_s : vertex set incident with E_S edges.
    for all $u \in V_s$**do**
        remove*u* from$G\{i.e.\ G - V_s\}$
    end

Componentnumber ← **find** the number of components of $G - V_S$
MaxCompVertexnum ← **find** the vertex number of maximum component of $G - V_S$
findENR←{Component number – number of (E_S) – MaxCompVertexnum }
end

4. CONCLUSION

In this study, we investigate the edge-neighbor rupture degree of graphs obtained by graph operations. The graph operations are used to obtain new graphs. Union, join and mostly thorny operations are taken into consideration in this work. These operations are performed to various graphs and their edge-neighbor rupture degrees were determined.

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