Code of the multidimensional fractional pseudo-Newton method using recursive programming

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Abstract

The following paper presents one way to define and classify the fractional pseudo-Newton method through a group of fractional matrix operators, as well as a code written in recursive programming to implement this method, which through minor modifications, can be implemented in any fractional fixed-point method that allows solving nonlinear algebraic equation systems.

Keywords: Fractional Operators; Group Theory; Fractional Iterative Methods; Recursive Programming.

1. Fractional Pseudo-Newton Method

To begin this section, it is necessary to mention that due to the large number of fractional operators that may exist [1–6], some sets must be defined to fully characterize the **fractional pseudo-Newton method**[1] [7–10]. It is worth mentioning that characterizing elements of fractional calculus through sets is the main idea behind of the methodology known as **fractional calculus of sets** [11]. So, considering a scalar function $h:\mathbb{R}^m \to \mathbb{R}$ and the canonical basis of $\mathbb{R}^m$ denoted by $\{\hat{e}_k\}_{k\geq 1}$, it is possible to define the following fractional operator of order α using Einstein notation

$$o_x^\alpha h(x) := \hat{e}_k o_k^\alpha h(x). \tag{1}$$

Therefore, denoting by ∂_k^n the partial derivative of order n applied with respect to the k-th component of the vector x, using the previous operator it is possible to define the following set of fractional operators

$$\mathrm{O}_{x,\alpha}^n(h) := \left\{ o_x^\alpha \ : \ \exists o_k^\alpha h(x) \ \text{ and } \ \lim_{\alpha\to n} o_k^\alpha h(x) = \partial_k^n h(x) \ \forall k\geq 1 \right\}, \tag{2}$$

whose complement may be defined as follows

$$\mathrm{O}_{x,\alpha}^{n,c}(h) := \left\{ o_x^\alpha \ : \ \exists o_k^\alpha h(x) \ \forall k\geq 1 \ \text{ and } \ \lim_{\alpha\to n} o_k^\alpha h(x) \neq \partial_k^n h(x) \ \text{ in at least one value } k\geq 1 \right\}, \tag{3}$$

as a consequence, it is possible to define the following set

$$\mathrm{O}_{c,x,\alpha}^{n,u}(h) := \left(\mathrm{O}_{x,\alpha}^n(h) \cup \mathrm{O}_{x,\alpha}^{n,c}(h) \right) \cap \left\{ o_x^\alpha \ : \ o_k^\alpha c \neq 0 \ \forall c \in \mathbb{R}\setminus\{0\} \ \text{ and } \ \forall k\geq 1 \right\}. \tag{4}$$

On the other hand, considering a constant function $h:\Omega\subset\mathbb{R}^m \to \mathbb{R}^m$, it is possible to define the following set

$$ {}_m\mathrm{O}_{c,x,\alpha}^{n,u}(h) := \left\{ o_x^\alpha \ : \ o_x^\alpha \in \mathrm{O}_{c,x,\alpha}^{n,u}([h]_k) \ \forall k \leq m \right\}, \tag{5}$$

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[1]Método pseudo-Newton fraccional.

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where $[h]_k : \Omega \subset \mathbb{R}^m \to \mathbb{R}$ denotes the k-th component of the function h. So, it is possible to define the following set of fractional operators

$$
{}_m\mathrm{MO}^{\infty,u}_{c,x,\alpha}(h) := \bigcap_{k\in\mathbb{Z}} {}_m\mathrm{O}^{k,u}_{c,x,\alpha}(h), \tag{6}
$$

which under the classical Hadamard product it is fulfilled that

$$
o_x^0 \circ h(x) := h(x)\ \forall o_x^\alpha \in {}_m\mathrm{MO}^{\infty,u}_{c,x,\alpha}(h). \tag{7}
$$

Considering that when using the classical Hadamard product in general $o_x^{p\alpha} \circ o_x^{q\alpha} \neq o_x^{(p+q)\alpha}$. It is possible to define the following modified Hadamard product [11]:

$$
o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha} := \begin{cases} o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha}, & \text{if } i \neq j \text{ (Hadamard product of type horizontal)} \\ o_{i,x}^{(p+q)\alpha}, & \text{if } i = j \text{ (Hadamard product of type vertical)} \end{cases}, \tag{8}
$$

and considering that for each operator o_x^α it is possible to define the following **fractional matrix operator**

$$
A_\alpha(o_x^\alpha) = \left([A_\alpha(o_x^\alpha)]_{jk}\right) = \left(o_k^\alpha\right), \tag{9}
$$

it is possible to obtain the following theorem:

Theorem 1. *Let o_x^α be a fractional operator such that $o_x^\alpha \in {}_m\mathrm{MO}^{\infty,u}_{c,x,\alpha}(h)$. So, considering the modified Hadamard product given by* (8)*, it is possible to define the following set of fractional matrix operator*

$$
{}_m\mathrm{G}(A_\alpha(o_x^\alpha)) := \left\{A_\alpha^{\circ r} = A_\alpha(o_x^{r\alpha}) \ :\ r \in \mathbb{Z} \ \text{ and } \ A_\alpha^{\circ r} = \left([A_\alpha^{\circ r}]_{jk}\right) := \left(o_k^{r\alpha}\right)\right\}, \tag{10}
$$

which corresponds to the Abelian group generated by the operator $A_\alpha(o_x^\alpha)$.

Proof. It should be noted that due to the way the set (10) is defined, just the Hadamard product of type vertical is applied among its elements. So, $\forall A_\alpha^{\circ p}, A_\alpha^{\circ q} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha))$ it is fulfilled that

$$
A_\alpha^{\circ p} \circ A_\alpha^{\circ q} = \left([A_\alpha^{\circ p}]_{jk}\right) \circ \left([A_\alpha^{\circ q}]_{jk}\right) = \left(o_k^{(p+q)\alpha}\right) = \left([A_\alpha^{\circ(p+q)}]_{jk}\right) = A_\alpha^{\circ(p+q)}, \tag{11}
$$

with which it is possible to prove that the set (10) fulfills the following properties, which correspond to the properties of an Abelian group:

$$
\begin{cases}
\forall A_\alpha^{\circ p}, A_\alpha^{\circ p}, A_\alpha^{\circ r} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \ \text{ it is fulfilled that } \left(A_\alpha^{\circ p} \circ A_\alpha^{\circ q}\right) \circ A_\alpha^{\circ r} = A_\alpha^{\circ p} \circ \left(A_\alpha^{\circ q} \circ A_\alpha^{\circ r}\right) \\
\exists A_\alpha^{\circ 0} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \ \text{ such that } \ \forall A_\alpha^{\circ p} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \ \text{ it is fulfilled that } \ A_\alpha^{\circ 0} \circ A_\alpha^{\circ p} = A_\alpha^{\circ p} \\
\forall A_\alpha^{\circ p} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \ \exists A_\alpha^{\circ -p} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \ \text{ such that } \ A_\alpha^{\circ p} \circ A_\alpha^{\circ -p} = A_\alpha^{\circ 0} \\
\forall A_\alpha^{\circ p}, A_\alpha^{\circ q} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \ \text{ it is fulfilled that } \ A_\alpha^{\circ p} \circ A_\alpha^{\circ q} = A_\alpha^{\circ q} \circ A_\alpha^{\circ p}
\end{cases}. \tag{12}
$$

□

From the previous theorem, it is possible to define the following group of fractional matrix operators [11]:

$$
{}_m\mathrm{G}_{FPN}(\alpha) := \bigcup_{o_x^\alpha \in {}_m\mathrm{MO}^{\infty,u}_{c,x,\alpha}(h)} {}_m\mathrm{G}(A_\alpha(o_x^\alpha)), \tag{13}
$$

where $\forall A_{i,\alpha}^{\circ p}, A_{j,\alpha}^{\circ q} \in {}_m\mathrm{G}_{FPN}(\alpha)$, with $i \neq j$, the following property is defined

$$A_{i,\alpha}^{\circ p} \circ A_{j,\alpha}^{\circ q} = A_{k,\alpha}^{\circ 1} := A_{k,\alpha}\left(o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha}\right), \quad p,q \in \mathbb{Z} \setminus \{0\}, \tag{14}$$

as a consequence, it is fulfilled that

$$\forall A_{k,\alpha}^{\circ 1} \in {}_m\mathrm{G}_{FPN}(\alpha) \text{ such that } A_{k,\alpha}\left(o_{k,x}^{\alpha}\right) = A_{k,\alpha}\left(o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha}\right) \exists A_{k,\alpha}^{\circ r} = A_{k,\alpha}^{\circ(r-1)} \circ A_{k,\alpha}^{\circ 1} = A_{k,\alpha}\left(o_{i,x}^{rp\alpha} \circ o_{j,x}^{rq\alpha}\right). \tag{15}$$

Then, it is possible to obtain the following result:

$$\forall A_{\alpha}^{\circ 1} \in {}_m\mathrm{G}_{FPN}(\alpha) \, \exists A_{\epsilon,\alpha} := A_{\alpha}^{\circ 1} \circ I_m + \epsilon I_m, \tag{16}$$

where I_m denotes the identity matrix of $m \times m$ and ϵ is a positive constant $\ll 1$. So, defining the following function

$$\beta(\alpha,[x]_k) := \begin{cases} \alpha, & \text{if} \quad |[x]_k| \neq 0 \\ 1, & \text{if} \quad |[x]_k| = 0 \end{cases}, \tag{17}$$

the fractional pseudo-Newton method may be defined and classified through the following set of matrices:

$$\left\{A_{\epsilon,\beta} = A_{\epsilon,\beta}\left(A_{\alpha}^{\circ 1}\right) \,:\, A_{\alpha}^{\circ 1} \in {}_m G_{FPN}(\alpha) \text{ and } A_{\epsilon,\beta}(x) = \left([A_{\epsilon,\beta}]_{jk}(x)\right)\right\}. \tag{18}$$

Therefore, if Φ_{FPN} denotes the iteration function of the fractional pseudo-Newton method, it is possible to obtain the following result:

$$\text{Let } \alpha_0 \in \mathbb{R} \setminus \mathbb{Z} \Rightarrow \forall A_{\alpha_0}^{\circ 1} \in {}_m\mathrm{G}_{FPN}(\alpha) \, \exists \Phi_{FPN} = \Phi_{FPN}(A_{\alpha_0}) \therefore \forall A_{\alpha_0} \, \exists \{\Phi_{FPN}(A_{\alpha}) \,:\, \alpha \in \mathbb{R} \setminus \mathbb{Z}\}. \tag{19}$$

To end this section, it is worth mentioning that the fractional pseudo-Newton method has been used in the study for the construction of hybrid solar receivers [7,8,12], and that in recent years there has been a growing interest in fractional operators and their properties for solving nonlinear algebraic equation systems [13–22].

2. Programming Code of Fractional Pseudo-Newton Method

The following code was implemented in Python 3 and requires the following packages:

```python
import math as mt
import numpy as np
from numpy import linalg as la
```

For simplicity, a two-dimensional vector function is used to implement the code, that is, $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, which may be denoted as follows:

$$f(x) = \begin{pmatrix} [f]_1(x) \\ [f]_2(x) \end{pmatrix}, \tag{20}$$

where $[f]_i : \Omega \subset \mathbb{R}^2 \to \mathbb{R} \; \forall i \in \{1,2\}$. Then considering a function $\Phi : (\mathbb{R} \setminus \mathbb{Z}) \times \mathbb{C}^n \to \mathbb{C}^n$, the fractional pseudo-Newton method may be denoted as follows [11,23]:

$$x_{i+1} := \Phi(\alpha, x_i) = x_i - A_{\epsilon,\beta}(x_i) f(x_i), \quad i = 0,1,2\cdots, \tag{21}$$

where $A_{\epsilon,\beta}(x_i)$ is a matrix evaluated in the value x_i, which is given by the following expression

$$A_{\epsilon,\beta}(x_i) = \left([A_{\epsilon,\beta}]_{jk}(x_i)\right) := \left(o_k^{\beta(\alpha,[x_i]_k)} \delta_{jk} + \epsilon \delta_{jk}\right)_{x_i}, \tag{22}$$

with δ_{jk} the Kronecker delta. It is worth mentioning that one of the main advantages of fractional iterative methods is that the initial condition x_0 can remain fixed, with which it is enough to vary the order α of the fractional operators involved until generating a sequence convergent $\{x_i\}_{i\geq 1}$ to the value $\xi \in \Omega$. Since the order α of the fractional operators is varied, different values of α can generate different convergent sequences to the same value ξ but with a different number of iterations. So, it is possible to define the following set

$$\mathrm{Conv}_{\delta}(\xi) := \left\{\Phi \ : \ \lim_{x\to\xi} \Phi(\alpha,x) = \xi_\alpha \in B(\xi;\delta)\right\}, \tag{23}$$

which may be interpreted as the set of fractional fixed-point methods that define a convergent sequence $\{x_i\}_{i\geq 1}$ to some value $\xi_\alpha \in B(\xi;\delta)$. So, denoting by card$(\cdot)$ the cardinality of a set, under certain conditions it is possible to prove the following result (see reference [11], proof of **Theorem** 2):

$$\mathrm{card}\left(\mathrm{Conv}_{\delta}(\xi)\right) = \mathrm{card}\left(\mathbb{R}\right), \tag{24}$$

from which it follows that the set (23) is generated by an uncountable family of fractional fixed-point methods. Before continuing, it is necessary to define the following corollary [11]:

Corollary 1. *Let $\Phi : (\mathbb{R} \setminus \mathbb{Z}) \times \mathbb{C}^n \to \mathbb{C}^n$ be an iteration function such that $\Phi \in \mathrm{Conv}_{\delta}(\xi)$. So, if Φ has an order of convergence of order (at least) p in $B(\xi;1/2)$, for some $m \in \mathbb{N}$, there exists a sequence $\{P_i\}_{i\geq m} \in B(p;\delta_K)$ given by the following values*

$$P_i = \frac{\log\left(\|x_i - x_{i-1}\|\right)}{\log\left(\|x_{i-1} - x_{i-2}\|\right)}, \tag{25}$$

such that it fulfills the following condition:

$$\lim_{i\to\infty} P_i \to p,$$

and therefore, there exists at least one value $k \geq m$ such that

$$P_k \in B(p;\epsilon). \tag{26}$$

The previous corollary allows estimating numerically the order of convergence of an iteration function Φ that generates at least one convergent sequence $\{x_i\}_{i\geq 1}$. On the other hand, the following corollary allows characterizing the order of convergence of an iteration function Φ through its **Jacobian matrix** $\Phi^{(1)}$ [11, 22]:

Corollary 2. *Let $\Phi : (\mathbb{R} \setminus \mathbb{Z}) \times \mathbb{C}^n \to \mathbb{C}^n$ be an iteration such that $\Phi \in \mathrm{Conv}_{\delta}(\xi)$. So, if Φ has an order of convergence of order (at least) p in $B(\xi;\delta)$, it is fulfilled that:*

$$p := \begin{cases} 1, & \text{if } \lim_{x\to\xi} \left\|\Phi^{(1)}(\alpha,x)\right\| \neq 0 \\ 2, & \text{if } \lim_{x\to\xi} \left\|\Phi^{(1)}(\alpha,x)\right\| = 0 \end{cases}. \tag{27}$$

Before continuing, it is necessary to mention that what is shown below is an extremely simplified way of how a fractional iterative method should be implemented. A more detailed description, as well as some applications, may be found in the references [11,20–23]. Considering the particular case with $\Phi : (\mathbb{R}\setminus\mathbb{Z})\times\mathbb{R}^n \to \mathbb{R}^n$, and defining the following notation:

$$ErrDom := \left\{\|x_i - x_{i-1}\|_2\right\}_{i\geq 1}, \quad ErrIm := \left\{\|f(x_i)\|_2\right\}_{i\geq 1}, \quad X := \left\{x_i\right\}_{i\geq 1}, \tag{28}$$

it is possible to implement a particular case of the multidimensional fractional pseudo-Newton method through recursive programming using the following functions [10]:

```
def Dfrac(α,x):
    return pow(x,-α)/mt.gamma(1-α) if abs(1-α)>0 else 0

def β(α,x):
    return α if abs(x)>0 else 1

def Aϵβ(α,x):
    N=len(x)
    y=np.zeros((N,N))
    ϵ=pow(10,-4)
    for i in range(0,N):
        y[i][i]=Dfrac(β(α,x[i]),x[i])+ϵ
    return y

def FractionalPseudoNewton(ErrDom,ErrIm,X,α,x0):
    Tol=pow(10,-5)
    Lim=pow(10,2)

    x1=x0-np.matmul(Aϵβ(α,x0),f(x0))
    ED=la.norm(x1-x0)

    if ED>0:
        EI=la.norm(f(x1))

        ErrDom.append(ED)
        ErrIm.append(EI)
        X.append(x1)
        N=len(X)

        if max(ED,EI)>Tol and N<Lim:
            ErrDom,ErrIm,X=FractionalPseudoNewton(ErrDom,ErrIm,X,α,x1)

    return ErrDom,ErrIm,X
```

To implement the above functions, it is necessary to follow the steps shown below:

i) A function must be programmed (information of the following nonlinear function may be found in the reference [9]).

```
def f(x):
    y=np.zeros((2,1))

    a1=0.5355
    a2=1.5808
    a3=1.5355
    a4=0.5808
    a5=18.9753
    a6=451474
    a7=396499

    d1=pow(x[0],a3)-pow(x[1],a3)
    d2=pow(x[0],a4)-pow(x[1],a4)
    d3=pow(x[0],a3+a4)-pow(x[1],a3+a4)

    y[0]=x[0]-(a6/a5)+(a2*x[0]*pow(x[1],a3)*d2-a1*pow(x[0],a2)*d1)/(a1*a2*d3)
    y[1]=x[1]-(a7/a5)+(a2*pow(x[0],a3)*x[1]*d2-a1*pow(x[1],a2)*d1)/(a1*a2*d3)
    return y
```

ii) Three empty vectors, a fractional order α, and an initial condition x_0 must be defined before implementing the function FractionalPseudoNewton.

```
ErrDom=[]
ErrIm=[]
X=[]

α=-0.02705

x0=np.ones((2,1))
x0[0]=1
x0[1]=2

ErrDom,ErrIm,X=FractionalPseudoNewton(ErrDom,ErrIm,X,α,x0)
```

When implementing the previous steps, if the fractional order α and initial condition x_0 are adequate to approach a zero of the function f, results analogous to the following are obtained:

i	$[x_i]_1$	$[x_i]_2$	$\|x_i - x_{i-1}\|_2$	$\|f(x_i)\|_2$
1	24154.6890055726	21615.770565224655	32412.27808445575	3173.9427518435878
2	23797.525207771814	17409.867461022026	4221.040973551523	2457.2339691838274
3	27022.686583015326	16837.96263298479	3275.4756950243	1936.4252930355906
4	28968.497786158376	15149.13086241385	2576.4964559585133	1988.3049656277824
5	31513.395908759314	14371.120308833728	2661.1664502431686	1670.221737448303
6	33594.7029990163	13550.005464416314	2237.4245890480047	1489.2609462571957
⋮	⋮	⋮	⋮	⋮
61	41844.57086184946	11857.321286205206	$6.11497121228039e-05$	$2.0557739187213006e-05$
62	41844.5708629114	11857.321259325998	$2.690017756661858e-05$	$2.567920334469916e-05$
63	41844.57089334319	11857.321275572	$3.449675719137571e-05$	$1.2220011320189923e-05$
64	41844.57089188268	11857.32125964514	$1.5993685206416137e-05$	$1.4647204650186194e-05$
65	41844.57090877583	11857.3212696822	$1.964996494386499e-05$	$7.37448238916247e-06$
66	41844.57090683372	11857.32126021737	$9.662028743039773e-06$	$8.428415184912125e-06$

Table 1: Results obtained using the fractional pseudo-Newton method [10].

Therefore, from the **Corollary 1**, the following result is obtained:

$$P_{66} = \frac{\log\left(\|x_{66} - x_{65}\|\right)}{\log\left(\|x_{65} - x_{64}\|\right)} \approx 1.0655 \in B(p;\delta_K),$$

which is consistent with the **Corollary 2**, since if $\Phi_{FPN} \in \text{Conv}_\delta(\xi)$, in general Φ_{FPN} fulfills the following condition (see reference [22], proof of **Proposition 1**):

$$\lim_{x\to\xi}\left\|\Phi_{FPN}^{(1)}(\alpha,x)\right\| \neq 0, \tag{29}$$

from which it is concluded that the fractional pseudo-Newton method has an order of convergence (at least) linear in $B(\xi;\delta)$.

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