Code of the multidimensional fractional quasi-Newton method using recursive programming

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Abstract

The following paper presents one way to define and classify the fractional quasi-Newton method through a group of fractional matrix operators, as well as a code written in recursive programming to implement this method, which through minor modifications, can be implemented in any fractional fixed-point method that allows solving nonlinear algebraic equation systems.

Keywords: Fractional Operators; Group Theory; Fractional Iterative Methods; Recursive Programming.

1. Fractional Quasi-Newton Method

To begin this section, it is necessary to mention that due to the large number of fractional operators that may exist [1–6], some sets must be defined to fully characterize the **fractional quasi-Newton method**[1] [7, 8]. It is worth mentioning that characterizing elements of fractional calculus through sets is the main idea behind of the methodology known as **fractional calculus of sets** [9]. So, considering a scalar function $h : \mathbb{R}^m \to \mathbb{R}$ and the canonical basis of $\mathbb{R}^m$ denoted by $\{\hat{e}_k\}_{k\geq 1}$, it is possible to define the following fractional operator of order α using Einstein notation

$$o_x^\alpha h(x) := \hat{e}_k o_k^\alpha h(x). \tag{1}$$

Therefore, denoting by ∂_k^n the partial derivative of order n applied with respect to the k-th component of the vector x, using the previous operator it is possible to define the following set of fractional operators

$$\mathrm{O}_{x,\alpha}^n(h) := \left\{ o_x^\alpha \ : \ \exists o_k^\alpha h(x) \ \text{ and } \ \lim_{\alpha\to n} o_k^\alpha h(x) = \partial_k^n h(x) \ \forall k \geq 1 \right\}, \tag{2}$$

whose complement may be defined as follows

$$\mathrm{O}_{x,\alpha}^{n,c}(h) := \left\{ o_x^\alpha \ : \ \exists o_k^\alpha h(x) \ \forall k \geq 1 \ \text{ and } \ \lim_{\alpha\to n} o_k^\alpha h(x) \neq \partial_k^n h(x) \ \text{ in at least one value } k \geq 1 \right\}, \tag{3}$$

as a consequence, it is possible to define the following set

$$\mathrm{O}_{c,x,\alpha}^{n,u}(h) := \left(\mathrm{O}_{x,\alpha}^n(h) \cup \mathrm{O}_{x,\alpha}^{n,c}(h) \right) \cap \left\{ o_x^\alpha \ : \ o_k^\alpha c \neq 0 \ \forall c \in \mathbb{R} \setminus \{0\} \ \text{ and } \ \forall k \geq 1 \right\}. \tag{4}$$

On the other hand, considering a linear function $h : \Omega \subset \mathbb{R}^m \to \mathbb{R}^m$, it is possible to define the following set

$$_m\mathrm{O}_{c,x,\alpha}^{n,u}(h) := \left\{ o_x^\alpha \ : \ o_x^\alpha \in \mathrm{O}_{c,x,\alpha}^{n,u}([h]_k) \ \forall k \leq m \right\}, \tag{5}$$

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[1]Método quasi-Newton fraccional.

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where $[h]_k : \Omega \subset \mathbb{R}^m \to \mathbb{R}$ denotes the k-th component of the function h. So, it is possible to define the following set of fractional operators

$$ {}_m\mathrm{MO}^{\infty,u}_{c,x,\alpha}(h) := \bigcap_{k\in\mathbb{Z}} {}_m\mathrm{O}^{k,u}_{c,x,\alpha}(h), \tag{6} $$

which under the classical Hadamard product it is fulfilled that

$$ o^0_x \circ h(x) := h(x)\ \forall o^\alpha_x \in {}_m\mathrm{MO}^{\infty,u}_{c,x,\alpha}(h). \tag{7} $$

Then, considering that for each operator o^α_x it is possible to define the following **fractional matrix operator**

$$ A_\alpha(o^\alpha_x) = \left([A_\alpha(o^\alpha_x)]_{jk}\right) = \left(o^\alpha_k\right), \tag{8} $$

it is possible to define the following set of fractional operators

$$ {}_m\mathrm{IMO}^{\infty,u}_{c,x,\alpha}(h) := {}_m\mathrm{MO}^{\infty,u}_{c,x,\alpha}(h) \cap \left\{ o^\alpha_x \ : \ \exists \left(A_\alpha\left(o^\alpha_x\right) \circ A^T_\alpha\left(h\right)\right)^{-1} \right\}, \tag{9} $$

where $A_\alpha(h) = \left([A_\alpha(h)]_{jk}\right) = \left([h]_k\right)$. On the other hand, considering that when using the classical Hadamard product in general $o^{p\alpha}_x \circ o^{q\alpha}_x \neq o^{(p+q)\alpha}_x$. It is possible to define the following modified Hadamard product [9]:

$$ o^{p\alpha}_{i,x} \circ o^{q\alpha}_{j,x} := \begin{cases} o^{p\alpha}_{i,x} \circ o^{q\alpha}_{j,x}, & \text{if } i \neq j \text{ (Hadamard product of type horizontal)} \\ o^{(p+q)\alpha}_{i,x}, & \text{if } i = j \text{ (Hadamard product of type vertical)} \end{cases}, \tag{10} $$

with which it is possible to obtain the following theorem:

Theorem 1. *Let o^α_x be a fractional operator such that $o^\alpha_x \in {}_m\mathrm{MO}^{\infty,u}_{c,x,\alpha}(h)$. So, considering the modified Hadamard product given by* (10), *it is possible to define the following set of fractional matrix operator*

$$ {}_m\mathrm{G}\left(A_\alpha\left(o^\alpha_x\right)\right) := \left\{ A^{\circ r}_\alpha = A_\alpha\left(o^{r\alpha}_x\right) \ : \ r \in \mathbb{Z} \ \text{ and } \ A^{\circ r}_\alpha = \left([A^{\circ r}_\alpha]_{jk}\right) := \left(o^{r\alpha}_k\right) \right\}, \tag{11} $$

which corresponds to the Abelian group generated by the operator $A_\alpha\left(o^\alpha_x\right)$.

Proof. It should be noted that due to the way the set (11) is defined, just the Hadamard product of type vertical is applied among its elements. So, $\forall A^{\circ p}_\alpha, A^{\circ q}_\alpha \in {}_m\mathrm{G}\left(A_\alpha\left(o^\alpha_x\right)\right)$ it is fulfilled that

$$ A^{\circ p}_\alpha \circ A^{\circ q}_\alpha = \left([A^{\circ p}_\alpha]_{jk}\right) \circ \left([A^{\circ q}_\alpha]_{jk}\right) = \left(o^{(p+q)\alpha}_k\right) = \left([A^{\circ(p+q)}_\alpha]_{jk}\right) = A^{\circ(p+q)}_\alpha, \tag{12} $$

with which it is possible to prove that the set (11) fulfills the following properties, which correspond to the properties of an Abelian group:

$$ \begin{cases} \forall A^{\circ p}_\alpha, A^{\circ p}_\alpha, A^{\circ r}_\alpha \in {}_m\mathrm{G}\left(A_\alpha\left(o^\alpha_x\right)\right) \ \text{ it is fulfilled that } \ \left(A^{\circ p}_\alpha \circ A^{\circ q}_\alpha\right) \circ A^{\circ r}_\alpha = A^{\circ p}_\alpha \circ \left(A^{\circ q}_\alpha \circ A^{\circ r}_\alpha\right) \\ \exists A^{\circ 0}_\alpha \in {}_m\mathrm{G}\left(A_\alpha\left(o^\alpha_x\right)\right) \ \text{ such that } \ \forall A^{\circ p}_\alpha \in {}_m\mathrm{G}\left(A_\alpha\left(o^\alpha_x\right)\right) \ \text{ it is fulfilled that } \ A^{\circ 0}_\alpha \circ A^{\circ p}_\alpha = A^{\circ p}_\alpha \\ \forall A^{\circ p}_\alpha \in {}_m\mathrm{G}\left(A_\alpha\left(o^\alpha_x\right)\right) \ \exists A^{\circ -p}_\alpha \in {}_m\mathrm{G}\left(A_\alpha\left(o^\alpha_x\right)\right) \ \text{ such that } \ A^{\circ p}_\alpha \circ A^{\circ -p}_\alpha = A^{\circ 0}_\alpha \\ \forall A^{\circ p}_\alpha, A^{\circ q}_\alpha \in {}_m\mathrm{G}\left(A_\alpha\left(o^\alpha_x\right)\right) \ \text{ it is fulfilled that } \ A^{\circ p}_\alpha \circ A^{\circ q}_\alpha = A^{\circ q}_\alpha \circ A^{\circ p}_\alpha \end{cases}. \tag{13} $$

□

From the previous theorem, it is possible to define the following group of fractional matrix operators [9]:

$$ {}_m\mathrm{G}_{FQN}(\alpha) := \bigcup_{o_x^\alpha \in_m \mathrm{IMO}_{c,x,\alpha}^{\infty,u}(h)} {}_m\mathrm{G}\left(A_\alpha\left(o_x^\alpha\right)\right), \tag{14} $$

where $\forall A_{i,\alpha}^{\circ p}, A_{j,\alpha}^{\circ q} \in {}_m\mathrm{G}_{FQN}(\alpha)$, with $i \neq j$, the following property is defined

$$ A_{i,\alpha}^{\circ p} \circ A_{j,\alpha}^{\circ q} = A_{k,\alpha}^{\circ 1} := A_{k,\alpha}\left(o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha}\right), \quad p,q \in \mathbb{Z} \setminus \{0\}, \tag{15} $$

as a consequence, it is fulfilled that

$$ \forall A_{k,\alpha}^{\circ 1} \in {}_m\mathrm{G}_{FQN}(\alpha) \text{ such that } A_{k,\alpha}\left(o_{k,x}^{\alpha}\right) = A_{k,\alpha}\left(o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha}\right) \exists A_{k,\alpha}^{\circ r} = A_{k,\alpha}^{\circ(r-1)} \circ A_{k,\alpha}^{\circ 1} = A_{k,\alpha}\left(o_{i,x}^{rp\alpha} \circ o_{j,x}^{rq\alpha}\right). \tag{16} $$

Then, it is possible to obtain the following result:

$$ \forall A_{\alpha}^{\circ 1} \in {}_m\mathrm{G}_{FQN}(\alpha)\ \exists A_{h,\alpha}^{-1} := A_\alpha\left(o_x^\alpha\right) \circ A_\alpha^T(h), \tag{17} $$

and defining the following function

$$ \beta(\alpha,[x]_k) := \begin{cases} \alpha, & \text{if } \quad \|[x]_k\| \neq 0 \\ 1, & \text{if } \quad \|[x]_k\| = 0 \end{cases}, \tag{18} $$

the fractional quasi-Newton method may be defined and classified through the following set of matrices:

$$ \left\{A_{h,\beta} = A_{h,\beta}\left(A_\alpha^{\circ 1}\right) : A_\alpha^{\circ 1} \in {}_m G_{FQN}(\alpha) \text{ and } A_{h,\beta}(x) = \left([A_{h,\beta}]_{jk}(x)\right)\right\}. \tag{19} $$

Therefore, if Φ_{FQN} denotes the iteration function of the fractional quasi-Newton method, it is possible to obtain the following result:

$$ \text{Let } \alpha_0 \in \mathbb{R}\setminus\mathbb{Z} \Rightarrow \forall A_{\alpha_0}^{\circ 1} \in {}_m\mathrm{G}_{FQN}(\alpha)\ \exists \Phi_{FQN} = \Phi_{FQN}(A_{\alpha_0}) \therefore \forall A_{\alpha_0}\ \exists\left\{\Phi_{FQN}(A_\alpha) : \alpha \in \mathbb{R}\setminus\mathbb{Z}\right\}. \tag{20} $$

To end this section, it is worth mentioning that the fractional quasi-Newton method has been used in the study for the construction of hybrid solar receivers [8], and that in recent years there has been a growing interest in fractional operators and their properties for solving nonlinear algebraic equation systems [7, 10–18].

2. Programming Code of Fractional Quasi-Newton Method

The following code was implemented in Python 3 and requires the following packages:

```python
import math as mt
import numpy as np
from numpy import linalg as la
```

For simplicity, a two-dimensional vector function is used to implement the code, that is, $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, which may be denoted as follows:

$$ f(x) = \begin{pmatrix} [f]_1(x) \\ [f]_2(x) \end{pmatrix}, \tag{21} $$

where $[f]_i : \Omega \subset \mathbb{R}^2 \to \mathbb{R}\ \forall i \in \{1,2\}$. Then considering a function $\Phi : (\mathbb{R}\setminus\mathbb{Z}) \times \mathbb{C}^n \to \mathbb{C}^n$, the fractional quasi-Newton method may be denoted as follows [8, 9]:

$$ x_{i+1} := \Phi(\alpha, x_i) = x_i - A_{g_f,\beta}(x_i) f(x_i), \quad i = 0,1,2\cdots, \tag{22} $$

where $A_{g_f,\beta}(x_i)$ is a matrix evaluated in the value x_i, which is given by the following expression

$$A_{g_f,\beta}(x_i) = \left([A_{g_f,\beta}]_{jk}(x_i)\right) := \left(o_k^{\beta(\alpha,[x_i]_k)}[g_f]_j(x)\right)_{x_i}^{-1}, \tag{23}$$

with $g_f(x) := f(x_i) + f^{(1)}(x_i)x$. It is worth mentioning that one of the main advantages of fractional iterative methods is that the initial condition x_0 can remain fixed, with which it is enough to vary the order α of the fractional operators involved until generating a sequence convergent $\{x_i\}_{i\geq 1}$ to the value $\xi \in \Omega$. Since the order α of the fractional operators is varied, different values of α can generate different convergent sequences to the same value ξ but with a different number of iterations. So, it is possible to define the following set

$$\mathrm{Conv}_\delta(\xi) := \left\{\Phi \ : \ \lim_{x\to\xi}\Phi(\alpha,x) = \xi_\alpha \in B(\xi;\delta)\right\}, \tag{24}$$

which may be interpreted as the set of fractional fixed-point methods that define a convergent sequence $\{x_i\}_{i\geq 1}$ to some value $\xi_\alpha \in B(\xi;\delta)$. So, denoting by card$(\cdot)$ the cardinality of a set, under certain conditions it is possible to prove the following result (see reference [9], proof of **Theorem** 2):

$$\mathrm{card}\left(\mathrm{Conv}_\delta(\xi)\right) = \mathrm{card}\left(\mathbb{R}\right), \tag{25}$$

from which it follows that the set (24) is generated by an uncountable family of fractional fixed-point methods. Before continuing, it is necessary to define the following corollary [9]:

Corollary 1. *Let $\Phi : (\mathbb{R}\setminus\mathbb{Z})\times\mathbb{C}^n \to \mathbb{C}^n$ be an iteration function such that $\Phi \in \mathrm{Conv}_\delta(\xi)$. So, if Φ has an order of convergence of order (at least) p in $B(\xi;1/2)$, for some $m \in \mathbb{N}$, there exists a sequence $\{P_i\}_{i\geq m} \in B(p;\delta_K)$ given by the following values*

$$P_i = \frac{\log\left(\|x_i - x_{i-1}\|\right)}{\log\left(\|x_{i-1} - x_{i-2}\|\right)}, \tag{26}$$

such that it fulfills the following condition:

$$\lim_{i\to\infty} P_i \to p,$$

and therefore, there exists at least one value $k \geq m$ such that

$$P_k \in B(p;\epsilon). \tag{27}$$

The previous corollary allows estimating numerically the order of convergence of an iteration function Φ that generates at least one convergent sequence $\{x_i\}_{i\geq 1}$. On the other hand, the following corollary allows characterizing the order of convergence of an iteration function Φ through its **Jacobian matrix** $\Phi^{(1)}$ [9,18]:

Corollary 2. *Let $\Phi : (\mathbb{R}\setminus\mathbb{Z})\times\mathbb{C}^n \to \mathbb{C}^n$ be an iteration such that $\Phi \in \mathrm{Conv}_\delta(\xi)$. So, if Φ has an order of convergence of order (at least) p in $B(\xi;\delta)$, it is fulfilled that:*

$$p := \begin{cases} 1, & \text{if } \lim_{x\to\xi}\left\|\Phi^{(1)}(\alpha,x)\right\| \neq 0 \\ 2, & \text{if } \lim_{x\to\xi}\left\|\Phi^{(1)}(\alpha,x)\right\| = 0 \end{cases}. \tag{28}$$

Before continuing it is necessary to mention that what is shown below is an extremely simplified way of how a fractional iterative method should be implemented, a more detailed description, as well as some applications, may be found in the references [7–9,17–19]. Considering the following notation:

$$ErrDom := \left\{\|x_i - x_{i-1}\|_2\right\}_{i\geq 1}, \quad ErrIm := \left\{\|f(x_i)\|_2\right\}_{i\geq 1}, \quad X := \left\{x_i\right\}_{i\geq 1}, \tag{29}$$

it is possible to implement a particular case of the multidimensional fractional quasi-Newton method through recursive programming using the following functions:

```
def Dfrac(α,μ,x):
    s=μ-α
    if μ>-1:
        return (mt.gamma(μ+1)/mt.gamma(s+1))*pow(complex(x),s) if mt.ceil(s)-s>0 or s>-1 else 0

def β(α,x):
    return α if abs(x)>0 else 1

def FractionalQuasiNewton(ErrDom,ErrIm,X,α,x0):
    Tol=pow(10,-5)
    Lim=pow(10,2)
    InvA=InvAgfβ(α,x0)

    if abs(la.det(InvA))>0:
        x1=x0-np.matmul(la.inv(InvA),f(x0))
        ED=la.norm(x1-x0)

        if ED>0:
            EI=la.norm(f(x1))

            ErrDom.append(ED)
            ErrIm.append(EI)
            X.append(x1)
            N=len(X)

            if max(ED,EI)>Tol and N<Lim:
                ErrDom,ErrIm,X=FractionalQuasiNewton(ErrDom,ErrIm,X,α,x1)

    return ErrDom,ErrIm,X
```

To implement the above functions, it is necessary to follow the steps shown below:

i) A function must be programmed together with its Jacobian matrix.

```
def f(x):
    y=np.zeros((2,1)).astype(complex)
    y[0]=np.sin(x[0])*pow(x[0],2)+ np.cos(x[1])*pow(x[1],3)-5
    y[1]=np.cos(x[0])*pow(x[0],3)-np.sin(x[1])*pow(x[1],2)-7
    return y

def Df(x):
    y=np.zeros((2,2)).astype(complex)
    y[0][0]=2*np.sin(x[0])*x[0]+np.cos(x[0])*pow(x[0],2)
    y[0][1]=3*np.cos(x[1])*pow(x[1],2)-np.sin(x[1])*pow(x[1],3)
    y[1][0]=3*np.cos(x[0])*pow(x[0],2)-np.sin(x[0])*pow(x[0],3)
    y[1][1]=-2*np.sin(x[1])*x[1]-np.cos(x[1])*pow(x[1],2)
    return y
```

ii) The matrix $A^{-1}_{g_f,\beta}$ must be programmed.

```
def InvAgfβ(α,x):
    f0=f(x)
    Df0=Df(x)

    g11=f0[0]
    g1x=Df0[0][0]
    g1y=Df0[0][1]

    g21=f0[1]
    g2x=Df0[1,0]
    g2y=Df0[1,1]

    β1=β(α,x[0])
    β2=β(α,x[1])

    y=np.zeros((2,2)).astype(complex)
    y[0][0]=(g11+g1y*x[1])*Dfrac(β1,0,x[0])+ g1x*Dfrac(β1,1,x[0])
    y[0][1]=(g11+g1x*x[0])*Dfrac(β2,0,x[1])+ g1y*Dfrac(β2,1,x[1])
    y[1][0]=(g21+g2y*x[1])*Dfrac(β1,0,x[0])+ g2x*Dfrac(β1,1,x[0])
    y[1][1]=(g21+g2x*x[0])*Dfrac(β2,0,x[1])+ g2y*Dfrac(β2,1,x[1])
    return y
```

iii) Three empty vectors, a fractional order α, and an initial condition x_0 must be defined before implementing the function FractionalQuasiNewton.

```
ErrDom=[]
ErrIm=[]
X=[]

α=-0.14154

x0=1.87*np.ones((2,1))

ErrDom,ErrIm,X=FractionalQuasiNewton(ErrDom,ErrIm,X,α,x0)
```

When implementing the previous steps, if the fractional order α and initial condition x_0 are adequate to approach a zero of the function f, results analogous to the following are obtained:

i	$[x_i]_1$	$[x_i]_2$	$\|x_i - x_{i-1}\|_2$	$\|f(x_i)\|_2$
1	2.253615105769526	1.1942449832449582	0.7770491567746352	15.554324370388906
2	−3.9004625603638927	7.157573208896508	8.569361437566664	240.708834467148
3	$-1.5541583533037069 - 1.1179838230007044i$	$4.4928672793164015 - 2.962300422113517e-15i$	3.722323058480951	27.692687964920736
4	$-1.944385547081847 - 1.0897773339058576i$	$4.940730623672673 - 0.13817042653542763i$	0.6105288779230554	29.219708233069287
5	$-1.8475774337627011 - 1.1801485625387642i$	$4.855392369520536 + 0.009503969164296189i$	0.21593775616146188	8.81592186304761
6	$-1.7960511251389712 - 1.250743007942112i$	$4.8220026103345015 + 0.10421795336899867i$	0.13313208112481426	5.23303705045184
⋮	⋮	⋮	⋮	⋮
41	$-1.76751483239573 - 1.2381890934335078i$	$4.798857098280522 + 0.06489785685468008i$	$9.746530597591146e-07$	$7.151481519961957e-05$
42	$-1.7675150566379345 - 1.2381893834982014i$	$4.798857321824474 + 0.06489811797226702i$	$5.025697814069152e-07$	$8.392901211001586e-05$
43	$-1.7675149484191435 - 1.2381895859611525i$	$4.798857156145471 + 0.06489836899944432i$	$3.783737954105193e-07$	$6.69477928230414e-05$
44	$-1.7675147384551197 - 1.2381895971905954i$	$4.798856853548587 + 0.06489842891980367i$	$3.7331798114472574e-07$	$3.9363864098046455e-05$
45	$-1.7675145927812685 - 1.2381894810754603i$	$4.798856620324174 + 0.0648983327382783i$	$3.1360501942818643e-07$	$1.4419381217770664e-05$
46	$-1.767514555643636 - 1.2381893489293105i$	$4.798856526732281 + 0.06489819878007232i$	$2.1341521058220815e-07$	$4.62494486008204e-06$

Table 1: Results obtained using the fractional quasi-Newton method [8].

Therefore, from the **Corollary 1**, the following result is obtained:

$$P_{46} = \frac{\log(\|x_{46} - x_{45}\|)}{\log(\|x_{45} - x_{44}\|)} \approx 1.0257 \in B(p; \delta_K),$$

which is consistent with the **Corollary 2**, since if $\Phi_{FQN} \in \text{Conv}_\delta(\xi)$, in general Φ_{FQN} fulfills the following condition (see reference [18], proof of **Proposition 1**):

$$\lim_{x \to \xi} \left\| \Phi_{FQN}^{(1)}(\alpha, x) \right\| \neq 0, \tag{30}$$

from which it is concluded that the fractional quasi-Newton method has an order of convergence (at least) linear in $B(\xi; \delta)$.

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