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Code of a multidimensional fractional quasi-Newton method with an order of convergence at least quadratic using recursive programming

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Abstract

The following paper presents a way to define and classify a family of fractional iterative methods through a group of fractional matrix operators, as well as a code written in recursive programming to implement a variant of the fractional quasi-Newton method, which through minor modifications, can be implemented in any fractional fixed-point method that allows solving nonlinear algebraic equation systems.

Keywords: Fractional Operators; Group Theory; Fractional Iterative Methods; Recursive Programming.

1. Fractional Quasi-Newton Method Accelerated

To begin this section, it is necessary to mention that due to the large number of fractional operators that may exist [1–13], some sets must be defined to fully characterize the **fractional quasi-Newton method accelerated**[1] [14,15]. It is worth mentioning that characterizing elements of fractional calculus through sets is the main idea behind of the methodology known as **fractional calculus of sets** [16]. So, considering a scalar function $h:\mathbb{R}^m \to \mathbb{R}$ and the canonical basis of $\mathbb{R}^m$ denoted by $\{\hat{e}_k\}_{k\geq 1}$, it is possible to define the following fractional operator of order α using Einstein notation

$$o_x^\alpha h(x) := \hat{e}_k o_k^\alpha h(x). \tag{1}$$

Therefore, denoting by ∂_k^n the partial derivative of order n applied with respect to the k-th component of the vector x, using the previous operator it is possible to define the following set of fractional operators

$$\mathrm{O}_{x,\alpha}^n(h) := \left\{ o_x^\alpha \ : \ \exists o_k^\alpha h(x) \ \text{ and } \ \lim_{\alpha\to n} o_k^\alpha h(x) = \partial_k^n h(x) \ \forall k \geq 1 \right\}, \tag{2}$$

whose complement may be defined as follows

$$\mathrm{O}_{x,\alpha}^{n,c}(h) := \left\{ o_x^\alpha \ : \ \exists o_k^\alpha h(x) \ \forall k \geq 1 \ \text{ and } \ \lim_{\alpha\to n} o_k^\alpha h(x) \neq \partial_k^n h(x) \ \text{ in at least one value } k \geq 1 \right\}, \tag{3}$$

as a consequence, it is possible to define the following set

$$\mathrm{O}_{x,\alpha}^{n,u}(h) := \mathrm{O}_{x,\alpha}^n(h) \cup \mathrm{O}_{x,\alpha}^{n,c}(h). \tag{4}$$

On the other hand, considering a function $h:\Omega \subset \mathbb{R}^m \to \mathbb{R}^m$, it is possible to define the following set

$$ {}_m\mathrm{O}_{x,\alpha}^{n,u}(h) := \left\{ o_x^\alpha \ : \ o_x^\alpha \in \mathrm{O}_{x,\alpha}^{n,u}([h]_k) \ \forall k \leq m \right\}, \tag{5}$$

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[1] Método quasi-Newton fraccional acelerado.

where $[h]_k : \Omega \subset \mathbb{R}^m \to \mathbb{R}$ denotes the k-th component of the function h. So, it is possible to define the following set of fractional operators

$$ {}_m\mathrm{MO}^{\infty,u}_{x,\alpha}(h) := \bigcap_{k\in\mathbb{Z}} {}_m\mathrm{O}^{k,u}_{x,\alpha}(h), \tag{6}$$

which under the classical Hadamard product it is fulfilled that

$$ o_x^0 \circ h(x) := h(x) \ \forall o_x^\alpha \in {}_m\mathrm{MO}^{\infty,u}_{x,\alpha}(h). \tag{7}$$

Then, considering that for each operator o_x^α it is possible to define the following **fractional matrix operator**

$$ A_\alpha\left(o_x^\alpha\right) = \left([A_\alpha(o_x^\alpha)]_{jk}\right) = \left(o_k^\alpha\right), \tag{8}$$

it is possible to define for each operator $o_x^\alpha \in {}_m\mathrm{MO}^{\infty,u}_{x,\alpha}(h)$ the following matrix

$$ A_{h,\alpha} := A_\alpha\left(o_x^\alpha\right) \circ A_\alpha^T(h), \tag{9}$$

where $A_\alpha(h) = \left([A_\alpha(h)]_{jk}\right) = \left([h]_k\right)$. On the other hand, considering that when using the classical Hadamard product in general $o_x^{p\alpha} \circ o_x^{q\alpha} \neq o_x^{(p+q)\alpha}$. It is possible to define the following modified Hadamard product [16]:

$$ o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha} := \begin{cases} o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha}, & \text{if } i \neq j \text{ (Hadamard product of type horizontal)} \\ o_{i,x}^{(p+q)\alpha}, & \text{if } i = j \text{ (Hadamard product of type vertical)} \end{cases}, \tag{10}$$

with which it is possible to obtain the following theorem:

Theorem 1. *Let o_x^α be a fractional operator such that $o_x^\alpha \in {}_m\mathrm{MO}^{\infty,u}_{x,\alpha}(h)$. So, considering the modified Hadamard product given by* (10)*, it is possible to define the following set of fractional matrix operators*

$$ {}_m\mathrm{G}\left(A_\alpha\left(o_x^\alpha\right)\right) := \left\{A_\alpha^{\circ r} = A_\alpha\left(o_x^{r\alpha}\right) \ : \ r \in \mathbb{Z} \ \text{ and } \ A_\alpha^{\circ r} = \left([A_\alpha^{\circ r}]_{jk}\right) := \left(o_k^{r\alpha}\right)\right\}, \tag{11}$$

which corresponds to the Abelian group generated by the operator $A_\alpha\left(o_x^\alpha\right)$.

Proof. It should be noted that due to the way the set (11) is defined, just the Hadamard product of type vertical is applied among its elements. So, $\forall A_\alpha^{\circ p}, A_\alpha^{\circ q} \in {}_m\mathrm{G}\left(A_\alpha\left(o_x^\alpha\right)\right)$ it is fulfilled that

$$ A_\alpha^{\circ p} \circ A_\alpha^{\circ q} = \left([A_\alpha^{\circ p}]_{jk}\right) \circ \left([A_\alpha^{\circ q}]_{jk}\right) = \left(o_k^{(p+q)\alpha}\right) = \left([A_\alpha^{\circ(p+q)}]_{jk}\right) = A_\alpha^{\circ(p+q)}, \tag{12}$$

with which it is possible to prove that the set (11) fulfills the following properties, which correspond to the properties of an Abelian group:

$$ \begin{cases} \forall A_\alpha^{\circ p}, A_\alpha^{\circ p}, A_\alpha^{\circ r} \in {}_m\mathrm{G}\left(A_\alpha\left(o_x^\alpha\right)\right) \ \text{ it is fulfilled that } \ \left(A_\alpha^{\circ p} \circ A_\alpha^{\circ q}\right) \circ A_\alpha^{\circ r} = A_\alpha^{\circ p} \circ \left(A_\alpha^{\circ q} \circ A_\alpha^{\circ r}\right) \\ \exists A_\alpha^{\circ 0} \in {}_m\mathrm{G}\left(A_\alpha\left(o_x^\alpha\right)\right) \ \text{ such that } \ \forall A_\alpha^{\circ p} \in {}_m\mathrm{G}\left(A_\alpha\left(o_x^\alpha\right)\right) \ \text{ it is fulfilled that } \ A_\alpha^{\circ 0} \circ A_\alpha^{\circ p} = A_\alpha^{\circ p} \\ \forall A_\alpha^{\circ p} \in {}_m\mathrm{G}\left(A_\alpha\left(o_x^\alpha\right)\right) \ \exists A_\alpha^{\circ -p} \in {}_m\mathrm{G}\left(A_\alpha\left(o_x^\alpha\right)\right) \ \text{ such that } \ A_\alpha^{\circ p} \circ A_\alpha^{\circ -p} = A_\alpha^{\circ 0} \\ \forall A_\alpha^{\circ p}, A_\alpha^{\circ q} \in {}_m\mathrm{G}\left(A_\alpha\left(o_x^\alpha\right)\right) \ \text{ it is fulfilled that } \ A_\alpha^{\circ p} \circ A_\alpha^{\circ q} = A_\alpha^{\circ q} \circ A_\alpha^{\circ p} \end{cases}. \tag{13}$$

From the previous theorem, it is possible to define the following group of fractional matrix operators [16]:

$$ {}_m\mathrm{G}_{FIM}(\alpha) := \bigcup_{o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)} {}_m\mathrm{G}\left(A_\alpha\left(o_x^\alpha\right)\right), \tag{14} $$

where $\forall A_{i,\alpha}^{\circ p}, A_{j,\alpha}^{\circ q} \in {}_m\mathrm{G}_{FIM}(\alpha)$, with $i \neq j$, the following property is defined

$$ A_{i,\alpha}^{\circ p} \circ A_{j,\alpha}^{\circ q} = A_{k,\alpha}^{\circ 1} := A_{k,\alpha}\left(o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha}\right), \quad p,q \in \mathbb{Z} \setminus \{0\}, \tag{15} $$

as a consequence, it is fulfilled that

$$ \forall A_{k,\alpha}^{\circ 1} \in {}_m\mathrm{G}_{FIM}(\alpha) \text{ such that } A_{k,\alpha}\left(o_{k,x}^{\alpha}\right) = A_{k,\alpha}\left(o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha}\right) \exists A_{k,\alpha}^{\circ r} = A_{k,\alpha}^{\circ(r-1)} \circ A_{k,\alpha}^{\circ 1} = A_{k,\alpha}\left(o_{i,x}^{rp\alpha} \circ o_{j,x}^{rq\alpha}\right). \tag{16} $$

Therefore, if Φ_{FIM} denotes the iteration function of some **fractional iterative method** [16], it is possible to obtain the following result:

$$ \text{Let } \alpha_0 \in \mathbb{R} \setminus \mathbb{Z} \Rightarrow \forall A_{\alpha_0}^{\circ 1} \in {}_m\mathrm{G}_{FIM}(\alpha) \, \exists \Phi_{FIM} = \Phi_{FIM}(A_{\alpha_0}) \therefore \forall A_{\alpha_0} \exists \{\Phi_{FIM}(A_\alpha) : \alpha \in \mathbb{R} \setminus \mathbb{Z}\}. \tag{17} $$

So, from the previous result, it is possible to define different sets that allow characterizing different fractional iterative methods. For example, the **fractional Newton-Raphson method** may be characterized through the following set [16,17]:

$$ {}_m\mathrm{G}_{FNR}(\alpha) := {}_m\mathrm{G}_{FIM}(\alpha) \cap \left\{ o_x^\alpha : \exists A_{h,\alpha}^{-1} = A_\alpha\left(o_x^\alpha\right) \circ A_\alpha^T(h) \right\}, \tag{18} $$

while the **fractional pseudo-Newton method** may be characterized through the following set [18,19]:

$$ {}_m\mathrm{G}_{FPN}(\alpha) := {}_m\mathrm{G}_{FIM}(\alpha) \cap \left\{ o_x^\alpha : o_k^\alpha c \neq 0 \; \forall c \in \mathbb{R} \setminus \{0\} \text{ and } \forall k \geq 1 \right\}, \tag{19} $$

as a consequence, the **fractional quasi-Newton method** may be characterized through the following set of fractional matrix operators [14,20]:

$$ {}_m\mathrm{G}_{FQN}(\alpha) := {}_m\mathrm{G}_{FNR}(\alpha) \cap {}_m\mathrm{G}_{FPN}(\alpha). \tag{20} $$

Before continuing it is necessary to define the following corollary:

Corollary 1. *Let $f : \Omega \subset \mathbb{R}^m \to \mathbb{R}^m$ be a function with a point $\xi \in \Omega$ such that $\|f(\xi)\| = 0$, and let $h : \Omega \subset \mathbb{R}^m \to \mathbb{R}^m$ be a function such that $h^{(1)}(x) = f^{(1)}(x) \; \forall x \in B(\xi;\delta)$. So, $\forall o_x^\alpha \in {}_m\mathrm{O}_{x,\alpha}^1(h)$ such that $A_\alpha\left(o_x^\alpha\right) \in {}_m\mathrm{G}_{FNR}(\alpha)$, there exists $A_{h,\alpha}^{-1} = A_\alpha\left(o_x^\alpha\right) \circ A_\alpha^T(h)$ such that it fulfills the following condition*

$$ \lim_{\alpha \to 1} A_{h,\alpha}(x) = \left(f^{(1)}(x)\right)^{-1} \; \forall x \in B(\xi;\delta). \tag{21} $$

Then, defining the following function

$$ \alpha_f([x]_k, x) := \begin{cases} \alpha, & \text{if } \; |[x]_k| \neq 0 \text{ and } \|f(x)\| > \delta_0 \\ 1, & \text{if } \; |[x]_k| = 0 \text{ or } \|f(x)\| \leq \delta_0 \end{cases}, \tag{22} $$

the fractional quasi-Newton method accelerated may be defined and classified through the following set of matrices [15]:

$$ \left\{ A_{h,\alpha_f} = A_{h,\alpha_f}\left(A_\alpha^{\circ 1}\right) : A_\alpha^{\circ 1} \in {}_m G_{FQN}(\alpha) \text{ and } \lim_{\alpha \to 1} A_{h,\alpha}(x) = \left(f^{(1)}(x)\right)^{-1} \text{ with } A_{h,\alpha_f}(x) = \left(\left[A_{h,\alpha_f}\right]_{jk}(x)\right) \right\}. \tag{23} $$

To end this section, it is worth mentioning that the fractional quasi-Newton method accelerated has been used in the study for the construction of hybrid solar receivers [15], and that in recent years there has been a growing interest in fractional operators and their properties for solving nonlinear algebraic equation systems [17,21–28].

2. Programming Code of Fractional Quasi-Newton Method Accelerated

The following code was implemented in Python 3 and requires the following packages:

```
import math as mt
import numpy as np
from numpy import linalg as la
```

For simplicity, a two-dimensional vector function is used to implement the code, that is, $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, which may be denoted as follows:

$$f(x) = \begin{pmatrix} [f]_1(x) \\ [f]_2(x) \end{pmatrix}, \tag{24}$$

where $[f]_i : \Omega \subset \mathbb{R}^2 \to \mathbb{R}\ \forall i \in \{1,2\}$. Then considering a function $\Phi : (\mathbb{R} \backslash \mathbb{Z}) \times \mathbb{C}^n \to \mathbb{C}^n$, a variant of the fractional quasi-Newton method may be denoted as follows [15, 16]:

$$x_{i+1} := \Phi(\alpha, x_i) = x_i - A_{h_f,\alpha_f}(x_i) f(x_i), \quad i = 0, 1, 2 \cdots, \tag{25}$$

where $A_{h_f,\alpha_f}(x_i)$ is a matrix evaluated in the value x_i, which is given by the following expression

$$A_{h_f,\alpha_f}(x_i) = \left([A_{h_f,\alpha_f}]_{jk}(x_i) \right) := \left(o_k^{\alpha_f([x_i]_k, x_i)} [h_f]_j(x) \right)_{x_i}^{-1}, \tag{26}$$

with $h_f(x) := f(x_i) + f^{(1)}(x_i)(x - x_i)$. It is worth mentioning that one of the main advantages of fractional iterative methods is that the initial condition x_0 can remain fixed, with which it is enough to vary the order α of the fractional operators involved until generating a sequence convergent $\{x_i\}_{i \geq 1}$ to the value $\xi \in \Omega$. Since the order α of the fractional operators is varied, different values of α can generate different convergent sequences to the same value ξ but with a different number of iterations. So, it is possible to define the following set

$$\mathrm{Conv}_\delta(\xi) := \left\{ \Phi \ : \ \lim_{x \to \xi} \Phi(\alpha, x) = \xi_\alpha \in B(\xi; \delta) \right\}, \tag{27}$$

which may be interpreted as the set of fractional fixed-point methods that define a convergent sequence $\{x_i\}_{i \geq 1}$ to some value $\xi_\alpha \in B(\xi; \delta)$. So, denoting by $\mathrm{card}(\cdot)$ the cardinality of a set, under certain conditions it is possible to prove the following result (see reference [16], proof of **Theorem 2**):

$$\mathrm{card}\left(\mathrm{Conv}_\delta(\xi)\right) = \mathrm{card}\left(\mathbb{R}\right), \tag{28}$$

from which it follows that the set (27) is generated by an uncountable family of fractional fixed-point methods. Before continuing, it is necessary to define the following corollary [16]:

Corollary 2. *Let $\Phi : (\mathbb{R} \setminus \mathbb{Z}) \times \mathbb{C}^n \to \mathbb{C}^n$ be an iteration function such that $\Phi \in \mathrm{Conv}_\delta(\xi)$. So, if Φ has an order of convergence of order (at least) p in $B(\xi; 1/2)$, for some $m \in \mathbb{N}$, there exists a sequence $\{P_i\}_{i \geq m} \in B(p; \delta_K)$ given by the following values*

$$P_i = \frac{\log\left(\|x_i - x_{i-1}\|\right)}{\log\left(\|x_{i-1} - x_{i-2}\|\right)}, \tag{29}$$

such that it fulfills the following condition:

$$\lim_{i \to \infty} P_i \to p,$$

and therefore, there exists at least one value $k \geq m$ such that

$$P_k \in B(p; \epsilon). \tag{30}$$

The previous corollary allows estimating numerically the order of convergence of an iteration function Φ that generates at least one convergent sequence $\{x_i\}_{i\geq 1}$. On the other hand, the following corollary allows characterizing the order of convergence of an iteration function Φ through its **Jacobian matrix** $\Phi^{(1)}$ [16, 28]:

Corollary 3. *Let $\Phi : (\mathbb{R} \setminus \mathbb{Z}) \times \mathbb{C}^n \to \mathbb{C}^n$ be an iteration such that $\Phi \in \text{Conv}_\delta(\xi)$. So, if Φ has an order of convergence of order (at least) p in $B(\xi;\delta)$, it is fulfilled that:*

$$p := \begin{cases} 1, & \text{if } \lim_{x\to\xi} \left\|\Phi^{(1)}(\alpha, x)\right\| \neq 0 \\ 2, & \text{if } \lim_{x\to\xi} \left\|\Phi^{(1)}(\alpha, x)\right\| = 0 \end{cases}. \tag{31}$$

Before continuing, it is necessary to mention that what is shown below is an extremely simplified way of how a fractional iterative method should be implemented, a more detailed description, as well as some applications, may be found in the references [14–18, 28–30]. Considering the following notation:

$$ErrDom := \left\{\|x_i - x_{i-1}\|_2\right\}_{i\geq 1}, \quad ErrIm := \left\{\|f(x_i)\|_2\right\}_{i\geq 1}, \quad X := \left\{x_i\right\}_{i\geq 1}, \tag{32}$$

it is possible to implement a particular case of the multidimensional fractional quasi-Newton method accelerated through recursive programming using the following functions:

```
def Dfrac(α,μ,x):
    s=μ-α
    if μ>-1:
        return (mt.gamma(μ+1)/mt.gamma(s+1))*pow(complex(x),s) if mt.ceil(s)-s>0 or s>-1 else 0

def αf(α,xk,normf):
    δ0=3
    return α if abs(xk)>0 and normf>δ0 else 1

def FractionalQuasiNewton(ErrDom,ErrIm,X,α,x0):
    Tol=pow(10,-5)
    Lim=pow(10,2)
    InvA=InvAhfαf(α,x0)

    if abs(la.det(InvA))>0:
        x1=x0-np.matmul(la.inv(InvA),f(x0))
        ED=la.norm(x1-x0)

        if ED>0:
            EI=la.norm(f(x1))

            ErrDom.append(ED)
            ErrIm.append(EI)
            X.append(x1)
            N=len(X)

            if max(ED,EI)>Tol and N<Lim:
                ErrDom,ErrIm,X=FractionalQuasiNewton(ErrDom,ErrIm,X,α,x1)

    return ErrDom,ErrIm,X
```

To implement the above functions, it is necessary to follow the steps shown below:

i) A function must be programmed together with its Jacobian matrix.

```
def f(x):
    y=np.zeros((2,1)).astype(complex)
    y[0]=np.sin(x[0])*pow(x[0],2)+ np.cos(x[1])*pow(x[1],3)-5
    y[1]=np.cos(x[0])*pow(x[0],3)-np.sin(x[1])*pow(x[1],2)-7
    return y

def Df(x):
    y=np.zeros((2,2)).astype(complex)
    y[0][0]=2*np.sin(x[0])*x[0]+np.cos(x[0])*pow(x[0],2)
    y[0][1]=3*np.cos(x[1])*pow(x[1],2)-np.sin(x[1])*pow(x[1],3)
    y[1][0]=3*np.cos(x[0])*pow(x[0],2)-np.sin(x[0])*pow(x[0],3)
    y[1][1]=-2*np.sin(x[1])*x[1]-np.cos(x[1])*pow(x[1],2)
    return y
```

ii) The matrix $A^{-1}_{h_f,\alpha_f}$ must be programmed.

```
def InvAhfαf(α,x):
    f0=f(x)
    Df0=Df(x)
    normf=la.norm(f0)

    h11=f0[0]
    h1x=Df0[0][0]
    h1y=Df0[0][1]

    h21=f0[1]
    h2x=Df0[1,0]
    h2y=Df0[1,1]

    α1=αf(α,x[0],normf)
    α2=αf(α,x[1],normf)

    y=np.zeros((2,2)).astype(complex)
    y[0][0]=(h11-h1x*x[0])*Dfrac(α1,0,x[0])+h1x*Dfrac(α1,1,x[0])
    y[0][1]=(h11-h1y*x[1])*Dfrac(α2,0,x[1])+h1y*Dfrac(α2,1,x[1])
    y[1][0]=(h21-h2x*x[0])*Dfrac(α1,0,x[0])+h2x*Dfrac(α1,1,x[0])
    y[1][1]=(h21-h2y*x[1])*Dfrac(α2,0,x[1])+h2y*Dfrac(α2,1,x[1])
    return y
```

iii) Three empty vectors, a fractional order α, and an initial condition x_0 must be defined before implementing the function FractionalQuasiNewton.

```
ErrDom=[]
ErrIm=[]
X=[]

α=-1.598394

x0=2.25*np.ones((2,1))

ErrDom,ErrIm,X=FractionalQuasiNewton(ErrDom,ErrIm,X,α,x0)
```

When implementing the previous steps, if the fractional order α and initial condition x_0 are adequate to approach a zero of the function f, results analogous to the following are obtained:

i	$[x_i]_1$	$[x_i]_2$	$\|x_i - x_{i-1}\|_2$	$\|f(x_i)\|_2$
1	-4.735585327165831	-1.0702683350651077	7.73450607213871	18.837153585624463
2	$-4.735751446234244 - 0.0005201775227302893i$	$-1.4523845601521614 - 1.1965409706379078i$	1.2560746005194463	15.550666339012087
3	$-4.737171940180769 - 0.0028188821384910173i$	$-1.3067374776351186 - 1.170476096763722i$	0.1479856484317521	18.23359009045263
4	$-4.738070885342333 - 0.004902921292654457i$	$-1.05437666853289 - 1.1528378083779465i$	0.2529866370176711	21.800921293297737
5	$-4.738147107048297 - 0.00657969269076015i$	$-0.5382886896737178 - 1.218835536081686i$	0.5202934934794708	25.01898906299888
⋮	⋮	⋮	⋮	⋮
12	$-4.729750257024641 - 0.004361695131642888i$	$0.5957441863423869 - 1.7216488909240522i$	0.00634398412485266	3.0045420137336363
13	$-4.7286434095025385 - 0.003971241380369586i$	$0.5950641918651102 - 1.724723641739149i$	0.0033606622313366525	2.9995701108099517
14	$-4.730316596597024 + 0.024479642825272506i$	$0.5937988516457116 - 1.7278144278472323i$	0.028695059013230096	0.05460293489272963
15	$-4.730869316300165 + 0.024410915392472112i$	$0.5939080267113869 - 1.7280910966350023i$	0.0006314169330564483	$2.0574246244311213e-05$
16	$-4.730869106529115 + 0.02441096901812828i$	$0.593907920270218 - 1.7280909090943382i$	$3.0558276730499713e-07$	$3.650221618055468e-12$

Table 1: Results obtained using the fractional quasi-Newton method accelerated [15].

Therefore, from the **Corollary 2**, the following result is obtained:

$$P_{16} = \frac{\log(\|x_{16} - x_{15}\|)}{\log(\|x_{15} - x_{14}\|)} \approx 2.0361 \in B(p;\delta_K),$$

which is consistent with the **Corollary 3**, since if $\Phi_{FQN} \in \mathrm{Conv}_\delta(\xi)$, in general $\Phi_{FQN}(A_{h_f,\alpha_f})$ fulfills the following condition (see reference [28], proof of **Proposition 1**):

$$\lim_{x \to \xi} \left\| \Phi^{(1)}_{FQN}(1,x) \right\| = 0, \tag{33}$$

from which it is concluded that the fractional quasi-Newton method accelerated has an order of convergence (at least) quadratic in $B(\xi;\delta)$.

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